1 Recap

In Lecture 18, we have talked about Linear Programming (LP). LP refers to the following problem. We are given an input of the following $m$ constraints (inequalities):

$$K \subset \mathbb{R}^n = \begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \leq b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \leq b_m \end{cases}$$

(1)

where $a_{ij}, b_i \in \mathbb{Q}$, for $i \in [m], j \in [n]$. Our goals are 1) to determine whether a solution exists, i.e., $K \neq \emptyset$ and 2) to maximize $c^T x$ for some $c \in \mathbb{Q}^n$ such that $x \in K$, if $K \neq \emptyset$. We will focus on goal 1). For goal 2), we can add one more constraint $c^T x \geq \text{opt}$ to $K$ and binary search for opt.

2 Solving Linear Programs

Actually, besides (1), LP also has other forms. Notice that for each $x_i$, we can rewrite it to $x_i = x_i^+ - x_i^-$ where $x_i^+, x_i^- \geq 0$. Therefore, after doing this transformation, (1) will become the following Standard Form:

$$K = \begin{cases} Ax \leq b \\ x \geq 0. \end{cases}$$

(2)

Let $A^{(i)}$ denote the $i$-th row of $A$. Then we can find that for each row $A^{(i)}$, $A^{(i)} \cdot x \leq b_i \iff A^{(i)} \cdot x + s_i = b_i, s_i \geq 0$. After adding $s_i$'s, (2) will become the following Equational Form:

$$K = \begin{cases} A'(x, s)^T = b \\ x \geq 0, s \geq 0 \end{cases} \text{ i.e. } K = \begin{cases} Ax = b \\ x \geq 0 \end{cases}$$

(3)

Therefore, it suffices to solve (3) in order to solve (1).
Definition 1. Suppose \( b = \frac{p}{q} \in \mathbb{Q} \) where \( p, q \in \mathbb{Z} \). Define \( |b| = |p| + |q| = \lceil \log_2|p| \rceil + \lceil \log_2|q| \rceil \).

Theorem 1. Given \( K \subset \mathbb{R}^n = \{ Ax = b, x \geq 0 \} \), let \( L = |A| + |b| = \sum_{ij} |a_{ij}| + \sum_i |b_i| \) be the size of input. If \( K \neq \emptyset \), there exists \( x \in K \) such that \( |x| = \text{poly}(L) \).

Proof. If we consider the geometric meanings of our constraints, then the set of solutions for \( Ax = b \) is an affine linear subspace in \( \mathbb{R}^n \). Suppose its dimension is \( m \). We will prove this theorem via mathematical induction on \( m \).

If \( m = 0 \), \( K \) is a single point and \( Ax = b \) gives a solution \( x \) in \( K \), which can be found in polynomial time by Gaussian Elimination. And this proves that \( |x| = \text{poly}(L) \).

If \( m = 1 \), \( Ax = b \) defines a line. The line can’t be parallel to all hyperplanes defined by \( x_i = 0 \). It means that there exists a \( i \in [n] \) such that \( \{ Ax = b, x_i = 0 \} \) gives a solution \( x \) with \( |x| = \text{poly}(L) \).

If \( m = 2 \), \( Ax = b \) defines a 2-dimensional hyperplane. Therefore, there exists \( i \in [n] \) such that \( x_i = 0 \) intersects with the plane. And \( \{ Ax = b, x_i = 0 \} \) defines a line. And we can use former approach when \( m = 1 \) to find a solution \( x \) with \( |x| = \text{poly}(L) \). Generally, for dimension \( m > 2 \), we can reduce it to the case when dimension is \( m - 1 \).

Corollary 2. Let \( K \subset \mathbb{R}^n = \{ Ax = b, x \geq 0 \} \). If \( K \neq \emptyset \), there exists a bounding box \( B = \{ x : \forall i, 0 \leq x_i \leq B_i \} \) such that \( K \cap B \neq \emptyset \) with \( |B| = \sum_i |B_i| = \text{poly}(L) \).

Definition 2. LP’ is the problem that given \( K \subset \mathbb{R}^n = \{ Ax \leq b, x_i \geq 0 \} \), our goal changes to the following: if \( \text{vol}(K) \geq V \), then output \( x \in K \) where \( V = 2^{-\text{poly}(L)} \) otherwise output nothing.

Theorem 3. If we can solve LP’ in polynomial time, we can also solve LP in polynomial time.

Remark 3.1. LP’ is a more relaxed (easier) task than LP.

Proof Sketch. If \( K = \emptyset \), let \( \gamma \) be the distance from \( \{ x : Ax = b \} \) to \( \{ x \geq 0 \} \). One can show that \( \gamma \geq 2^{-\text{poly}(L)} = \gamma_0 \). Let \( K' = \{ x : -\gamma_0 \leq Ax - b \leq \gamma_0, x \geq 0 \} \). We will get \( K' = \emptyset \).

If \( K \neq \emptyset \), \( K' \) has volume at least \( \left( \frac{\gamma_0}{2^{\text{poly}(L)}} \right)^n = 2^{-\text{poly}(L)} \). Therefore, if LP’ is solvable in polynomial time, we can relax \( K \) to \( K' \). After that, we can trigger the LP’ solver.

Here is a geometric view in \( \mathbb{R}^2 \).
2.1 The Ellipsoid Method for LP

In this subsection, we will introduce a concrete method for solving LP. The high level idea is that we maintain an ellipsoid that contains $K$. We can assume $\text{vol}(K) \geq 2^{-\text{poly}(L)} = V$ by Theorem 3. Here is the algorithm.

**Algorithm 1 The Ellipsoid Algorithm**

1: **Input:** $K \subset \mathbb{R}^n = \{ Ax \leq b \mid x \geq 0 \}$.
2: **Initialization:** Let $E_0$ be the smallest ellipsoid containing the bounding box $B$ defined in Corollary 2. Set $i = 1$.
3: **while TRUE do**
4: \hspace{1em} Let $x$ be the center of $E_{i-1}$.
5: \hspace{1em} if $x \in K$ then
6: \hspace{2em} **Output:** $x$.
7: \hspace{1em} else
8: \hspace{2em} Find an arbitrary linear constrain that is violated namely $a \cdot x \leq b$. Let $E_i$ be the smallest ellipsoid containing $E_{i-1} \cap \{ x : a \cdot x \leq b \}$.
9: \hspace{2em} if $\text{vol}(E_i) < V$ then
10: \hspace{3em} **Output:** NO SOLLUTION.
11: \hspace{2em} $i = i + 1$.

The following graph gives a geometric view of picking $E_1$. 

![Geometric View](image)

(a) $K = \emptyset$

(b) $K \neq \emptyset$

Figure 1: A geometric view in $\mathbb{R}^2$
If a solution is returned, it must be a feasible solution of $K$. If NO SOLUTION returned, $\text{vol}(K)$ must be $< V = 2^{-\text{poly}(L)}$ since $K \subset E_i$ for any $i$. Therefore, we only need to bound the number of iterations performed.

**Claim 4.** For every $i = 1, 2, 3, \ldots$, $$\frac{\text{vol}(E_i)}{\text{vol}(E_{i-1})} \leq 1 - \frac{1}{3n}.$$ 

**Proof.** Since every ellipsoid can be obtained via invertible linear transformations from a unit ball, and the volume is preserved upon a scaling factor (the determinant of the linear transformation matrix), we can assume w.l.o.g. that $E_{i-1}$ is the unit ball.

Now the worst case for the separating hyperplane is to go through the origin. Assume w.l.o.g. that it is $x_1 \geq 0$.

Let $E_i$ be centered at $(t, 0, \ldots, 0)$ ($t < 1/2$). Let the semi-axis along $x_1$ be $(1 - t)$ and semi-axis along other directions be $s$, i.e. $E_i$ satisfies the following equation:

$$\frac{(x_1 - t)^2}{(1-t)^2} + \frac{x_2^2}{s^2} + \cdots + \frac{x_n^2}{s^2} \leq 1.$$ 

To contain the half ball $E_{i-1} \cap \{x : x_1 \geq 0\}$ in $E_i$, we only need $(0, 1, 0, \ldots, 0) \in E_{i-1}$, i.e.

$$\frac{t^2}{(1-t)^2} + \frac{1}{s^2} \leq 1.$$ 

Here is the geometric view of above procedure.
From this inequality, we can get \( s^2 \geq \frac{(1-t)^2}{1-2t} \). Set \( s^2 = \frac{(1-t)^2}{1-2t} \) and \( t = \frac{1}{2n} \). We have

\[
\frac{\operatorname{vol}(E_i)}{\operatorname{vol}(E_{i-1})} = (1-t) \cdot s^{n-1}
\]

\[
= \left( \frac{(1-t)^2}{1-2t} \right)^{(n-1)/2} (1-t)
\]

\[
= \left( \frac{1 - 1/n + 1/(4n^2)}{1 - 1/n} \right)^{(n-1)/2} \left( 1 - \frac{1}{2n} \right)
\]

\[
= \left( 1 + \frac{1}{4n(n-1)} \right)^{(n-1)/2} \left( 1 - \frac{1}{2n} \right)
\]

\[
\leq \left( e^{\frac{4}{3n(n-1)}} \right)^{(n-1)/2} e^{-\frac{1}{2n}}
\]

\[
= e^{-\frac{3}{8n}}
\]

\[
\leq 1 - \frac{1}{3n}.
\]

In general, if \( E_i \) is not the unit ball, let \( \sigma \) be the invertible linear transformation such that \( \sigma(\text{unit ball}) = E_i \). Let \( \det(\sigma) \) be the determinant of the matrix associated with \( \sigma \). Let \( a' \cdot x \leq b' \) be the half-space obtained by \( \sigma^{-1}(\{a \cdot x \leq b\}) \). Let \( E'_i \) be the smallest ellipsoid containing unit ball \( \cap \{a' \cdot x \leq b'\} \). By previous discussion:

\[
\frac{\operatorname{vol}(E'_i)}{\operatorname{vol}(\text{unit ball})} \leq 1 - \frac{1}{3n}.
\]
Let $E_i = \sigma(E'_i)$. We know that $E_i$ contains $E_{i-1} \cap \{a \cdot x \leq b\}$ since $\sigma$ is bijective. Finally, we will have
\[
\frac{\text{vol}(E'_i)}{\text{vol}(	ext{unit ball})} = \frac{\det(\sigma) \cdot \text{vol}(E'_i)}{\det(\sigma) \cdot \text{vol}(	ext{unit ball})} \leq 1 - \frac{1}{3n}. \]

Claim 5. The Ellipsoid Algorithm terminates within $\text{poly}(L)$ iterations.

Proof. Suppose the algorithm does not terminate after $i$ iterations. Then we must have $\text{vol}(E_i) \geq V = 2^{-\text{poly}(L)}$. Also by selection of $E_0$, we have $\text{vol}(E_0) \leq 2^{O(n)} \cdot \text{vol}(B) \leq 2^{\text{poly}(L)}$. Further by claim 4, $\frac{\text{vol}(E_i)}{\text{vol}(E_0)} \leq \left(1 - \frac{1}{3n}\right)^i$. Therefore,
\[
i \leq \frac{\log(\text{vol}(E_i)/\text{vol}(E_0))}{\log(1 - \frac{1}{3n})} \leq O(n) \cdot \log(2^{\text{poly}(L)}/2^{\text{poly}(L)}) = O(n) \cdot \text{poly}(L) = \text{poly}(L).
\]

With above argument, we will get this algorithm terminates within $\text{poly}(L)$ iterations. \qed

Reference

b) [https://www.cs.cmu.edu/afs/cs.cmu.edu/academic/class/15859-f11/www/notes/lecture08.pdf](https://www.cs.cmu.edu/afs/cs.cmu.edu/academic/class/15859-f11/www/notes/lecture08.pdf)