Hardness of Approximation II  Simon Zhang

1. Unique games conjecture
2. Review: Goemans-Williamson for MAX-CUT
3. Unique hardness of MAX-CUT \(\frac{\cos \phi}{\pi} + 3,1 - \frac{\sqrt{3}}{8}\)

- reduction
- completeness

Past lecture: decision problem for MAX-E3LIN
\(\leq \frac{1}{2} + \varepsilon\) vs \(\geq 1 - \delta\) is W[1]-Hard

Notation: MAX-E3LIN \(\frac{1}{2} + \varepsilon, 1 - \delta\)

MAX-E3LIN \(\frac{1}{2} + \varepsilon, 1 - \delta\) \(\geq \rho\) LABEL-COVER \(\eta, 1 - \delta\)

Goal: today show

\(\text{MAX-CUT} \geq \frac{\cos \phi}{\pi} + 5,1 - \frac{\sqrt{3}}{8}\) \(\geq \rho\) ULC \(\eta, 1 - \delta\)

\(\Rightarrow\) Goemans-Williamson algorithm is optimal for MAX-CUT
Unique Games Conjecture

**Definition** Unique Label Cover (ULC(m))

- **Input** - Bipartite graph \((V \cup W, E)\)
- Set of labels \(\Sigma\), where \(|\Sigma| = m\)
- Edge functions: For all \((v, w) \in E\), there exists bijection \(\pi_{vw}: \Sigma \rightarrow \Sigma\)

Output: Label all vertices \(\sigma: V \cup W \rightarrow \Sigma\) such that the fraction of satisfied edges is maximized.

Satisfying edge \((v, w) \iff \pi_{vw}(\sigma(v)) = \sigma(w)\)

**Example** \(\Sigma = \{1, 2, 3\}\)

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1 \rightarrow 11, 12, 13
2 \rightarrow 21, 22
3 \rightarrow 31, 32
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All edges satisfied
Remark \([-\text{ULC} (\mu, \gamma) \text{ is in } \mathbb{P}.\]

Insight: Since \(\pi\) and \(\nu\) are bijections, given the label of a vertex, the labels of its neighbors are fixed. For each connected component of the bipartite graph, we can try all possible labels for a starting vertex and then search the rest of the component. This search is \(\text{poly}(m, 1, \nu, \mu, \gamma)\).

**Unique games conjecture (Khot 2002) (UGC)**

For all \(\delta > 0\), there exists \(m\) such that \(\text{ULC}(m, \gamma, 1-\delta)\) is \(\text{NP-hard}\).

Note: all we did is change \(\delta\) to \(1-\delta\) from the remark.

If conjecture is true, then a wide range of known approximations for optimization problems such as \(\text{MAX-CUT}, \text{MAX-2SAT}, \text{VERTEX-COVER}\) are all optimal!

We will show that Goemans-Williamson is optimal for \(\text{MAX-CUT}\), assuming UGC.
2. Review: Goemans-Williamson for MAX-CUT

\[ \text{MAX-CUT} \rightarrow \text{Relax} \rightarrow \text{SDP} \]

Let \( d_{\text{GW}} = \frac{2}{n} \min_{i \neq j} \frac{\cos \theta_{ij}}{1 - p} \times 0.879 \)

Then \( p_{\text{GW}} = -0.689 \)

Goemans-Williamson is \( d_{\text{GW}} \)-approx. for \( \text{MAX-CUT} \).

3. UG Hardness of MAX-CUT

Theorem: Let \( \varepsilon > 0 \), \(-1 < p < 0\). There exist \( \delta \) and \( m \) such that

\[ \text{MAX-CUT} \geq \frac{\cos \theta_{ij}}{1 - \varepsilon}, \quad \frac{1 - p - \varepsilon}{2} \geq p \cdot \text{ULC}(m) \]

Corollary: Goemans-Williamson is optimal, assuming UGC.

Proof: NP hard to approximate \( \text{MAX-CUT} \) factor

\[ 2 \geq \frac{\cos \theta_{ij}}{1 - \varepsilon} \geq d_{\text{GW}} \]

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Proof of Theorem 3 Parts: reduction, completeness.

Soundness

Next lecture

Pick $\delta$ sufficiently small (will specify how small (later) and then $m$ such that $ULC(m), \delta$ is NP-hard (assuming UGC).

We have black box that solves $\max\text{-}CUT$ for $\frac{1}{m+\varepsilon}, \frac{2\delta}{\varepsilon}, 2-2\delta$ and we have instance of $ULC(m), \delta$.

Let $G = (V_{UW}, E)$ be bipartite graph in $ULC(m)$.

Create new graph $H$ with vertex set $(V_{UW}) \times \{-1, 1\}^m$.

Associate to each $v \in V_{UW}$ a long code encoding $f_v: \{-1, 1\}^m \to \{-1, 1\}$ given a cut of $H$, vertices in one subset go to $-1$, and vertices in the other subset go to $1$.

Example $G$, $m = 2$
First try at reduction

Notation:
- \( x, y \in \{-1,1\}^m \)
- \( \Pi : [m] \to [m] \) bijection
- \( x \cdot y \) coordinate-wise multiplication: \( (xy)_i = x_i y_i \)
- \( x \cdot \Pi = (x\pi(1), \ldots, x\pi(m)) \)

1. Randomly pick \((v_i, w) \in E\) with permutation \(\Pi\)
2. Pick \( x \in \{-1,1\}^m \) randomly
3. Pick \( m \in \{-1,1\}^m \) independently pick each coordinate \(\pm 1\) w.p. \(\frac{1}{2}\)

4. Test \( f_v(x) \neq f_w((xy) \cdot \Pi) \)
   - if true, put edge between vertex in \( v \)'s hypercube labeled by \( x \) and vertex in \( w \)'s hypercube labeled by \((xy) \cdot \Pi\).

Resulting graph bipartite: could make all \( f_v \equiv 1 \) for \( v \in V \) and all \( f_w \equiv 1 \) for \( w \in W \) and test always passes!
Actual reduction only look at 2 long codes if V,W derives from one of V,W

1. Pick \( v \in V \) uniformly, and two edges \((v, w), (v, w')\)
   uniformly at random with permutations \( \pi, \pi' \), respectively.

2. Pick \( X \in \{-1, 1\}^m \) uniformly.

3. Pick \( \alpha \in \{-1, 1\} \) where \( \alpha \) coordinate \( \pi \)
   independently chosen as
   \[ M_i = \begin{cases} -1 & \text{w.p.} \frac{1}{2} \\ 1 & \text{w.p.} \frac{1}{2} \end{cases} \]

4. Test \( f_V(x \circ \pi) \neq f_{V'}(x \circ (x \pi) \circ \pi') \)

\( (v, w) \) satisfied \( \iff \pi' (\sigma(w)) = \sigma(v), \pi (\sigma(w)) = \sigma(v) \)
Completeness

Suppose \( ULC(m) \) has assignment \( \sigma : VUW \to \Sigma \)

satisfying \( \geq 1-\delta \) fraction of edges. Let each \( f \) be proper long encoding of \( \sigma(w) \).

The distance function: \( f_\nu(x) = x_\sigma(w) \) for all \( \nu \in VUW \), \( x \in \{-1,1\}^n \).

Edges \((v_1w)_1(v_2w)_2 \) from test

\[ A = (v_1w) \text{ satisfied} \]

\[ A' = (v_2w) \text{ satisfied} \]

By union bound,

\[ \Pr(\neg A \cup \neg A') = 1 - \Pr(\neg A \cup \neg A') \]

\[ \geq 1 - \Pr(\neg A) - \Pr(\neg A') \]

\[ \geq 1 - 2\delta \]

Now look at prob. test passes given \( A \cap A' \).

\[ f_\nu(x_{\sigma(w)}) = (x_{\sigma(w)})_\sigma(w) = x_{\pi(\sigma(w))} = x_{\sigma(v)} \]

\[ f'_{\nu'}((x_{\nu'}w)_1) = ((x_{\nu'}w)_1)_{\sigma(w')}' = (x_{\nu'}w)_{\sigma(w')} = (x_{\nu'}w)' = x_{\sigma(v)}M_{\sigma(v)} \]
\[ \Pr \left[ f_w(x_0 \pi) \neq f_{w_1}(x_0 \pi_{i_1}) \right] \]
\[ = \Pr \left[ \sigma(v) \neq \sigma(v) \cdot \sigma(v) \right] \]
\[ = \Pr \left[ \sigma(v) = -1 \right] \]
\[ = \frac{1-p}{2} \]

Thus,
\[ \Pr (\text{test passes}) \geq \left( \frac{1-p}{2} \right) (1 - 2\delta) \geq \left( \frac{1-p}{2} \right) \geq 3 \]
\[ \iff \delta \geq \frac{3}{2} \]

Shows completeness, next lecture will show soundness.