Spectral Graph Theory I - Christopher Ford 2/10/2015

Agenda: 1) Graph theory review and vertex functions
        2) Local variance
        3) Random walks & random vertices
        4) Inner products (time permitting)

Graph Theory Review

- A graph $G$ is defined by its vertices and edges.
  $G = (V, E)$ where $V$ is the set of vertices, $E$ the set of edges.

- For our graphs we assume the following:
  1) $V, E$ are finite $\Rightarrow G$ is finite
  2) no isolated vertices eg. $D_n$, $Y_n$
     (no vertices of degree zero)
  3) the edges do not have weights and are undirected
     (though we can augment our analysis w/ parallel edges to account for it)

Note: parallel edges and self-loops are allowed

Self-loops can be thought of as 1/2 edges and contribute 1 to the degree of the vertex they are connected to.

Note: degree = # edges adjacent to a vertex, if degrees of all vertices are the same, the graph is regular

Vertex Functions

- useful to label vertices w/ a function
  $f: V \rightarrow \mathbb{R}$
  eg. 6.046 exam problem, voltages, indicator

  indicator function: $0/1$ if $V_i \in S$, $S \subseteq V$

  $f: V \rightarrow \mathbb{R} = \begin{bmatrix} f(v_1) \\ f(v_2) \\ \vdots \\ f(v_n) \end{bmatrix}$

  can be thought of as a vector
Vector Functions Continued

Note: addition and scaling of these functions are preserved:

\[(f + g)(v_i) = f(v_i) + g(v_i)\]

\[(c \cdot f)(v_i) = c \cdot f(v_i)\]

\[\Rightarrow \text{linearity of functions}\]

Local Variance

** The main idea of spectral graph theory **

In simple terms:

We have a function \( f \) that maps \( V \rightarrow \mathbb{R} \). Want to know how much the function varies between vertices in the graph.

**Def:** local variance ("Dirichlet form", "analytic boundary size")

\[E(f) = \frac{1}{2} \sum_{u \sim v} [(f(u) - f(v))^2]\]

on \( UV \) probability dist. across edges

Immediate observations:

1) \( E(f) \geq 0 \)

2) \( E(cf) = c^2 E(f) \)

3) \( E(f + c) = E(f) \)

\#1 is trivial. \#2: \( E(cf) = \frac{1}{2} \sum_{u \sim v} [(cf(u) - cf(v))^2] \)

\[= \frac{1}{2} \sum_{u \sim v} [c(f(u) - f(v))^2] \]

\[= \frac{c^2}{2} \sum_{u \sim v} [(f(u) - f(v))^2] = c^2 E(f) \]

\#3: \( E(c + f) = \frac{1}{2} \sum_{u \sim v} [(f(u) + c - f(v) - c)^2] \)

\[= \frac{1}{2} \sum_{u \sim v} [(f(u) - f(v))^2] = E(f) \]
Intuition for local variance:
1. \( E(f) \) is small when \( f \) doesn't differ much at adjacent vertices. \( f \) is "smooth" across edges.
2. \( E(f) \) is large in the other case, \( f \) is "anti-smooth" or "rough" (not formal term).

Example (time permitting)
\[
f(v) = \begin{cases} 
1 & \text{if } v \in S \\
0 & \text{if } v \notin S 
\end{cases} 
\]

Indicator function
\[
f(v) = I_S(v)
\]
\[
E(f) = \frac{1}{2} \sum_{u \in V} ((I_S(u) - I_S(v))^2) \]
\[
= \frac{1}{2} \sum_{u \in V} [1((u,v) \text{ "crosses" } S) - 1((u,v) \text{ not in } S)]
\]
\[
= \Pr[u \to v \text{ "steps" out of } S \text{ subset}] 
\]

Random Vertices + Random Walks

\( P \) : distribution of randomly selected vertices chosen by the following procedure

1. Choose a random edge \( uv \)
2. Output \( u \) (or \( v \), identical by symmetry)

Think of \( P \) as a distribution across vertices weighted by their degree
more degree \( \Rightarrow \) more likely to be chosen in step 1.

\[
P[v] = \frac{\deg(v)}{|E|} \quad (\text{pick edge adj. to } u, \text{ output } u \text{ as endpoint})
\]
Example:

\[ \pi(v_1) = \pi(v_2) = \pi(v_3) = \frac{1}{3} \]

\[ \pi(v_4) = \pi(v_5) = \pi(v_6) = \pi(v_7) = \frac{1}{8} \]

\[ \pi(v_8) = \frac{4}{2 \cdot 4} = \frac{1}{2} \]

Application to random walks:

1. Picking u from \( \pi \) then picking v as a uniformly random neighbor of u is the same as

2. Drawing an edge uniformly at random \( u \sim v \)

\[ \text{prob of getting } uv \text{ from } (1) = \frac{\deg(u)}{2|E|} \cdot \frac{1}{\deg(v)} = \frac{1}{2|E|} = \text{prob of picking random edge} \]

Pick u, select v.

Procedure (1) is essentially a 1-step random walk. By repeating this step, we go on longer and longer walks. As we repeat, the distribution of the end pt. of our walk becomes \( \pi \).

Formally, let T.E.N. pick \( u \sim \pi \). Do a random walk starting @ \( u \) taking 1 steps. Distribution of v, the end pt. of the walk is \( \pi \).

\( \pi \) is also known as the stationary distribution on vertices. Also known as limiting/invariant distribution.

Suppose we don't start @ a random point. Say we start @ a pt. \( u_0 \).

Under what cases will the dist. of \( v \) not converge to \( \pi \)?

1. \( G \) is disconnected
2. \( G \) is bipartite (know which half you're in by even/odd steps)
When will it take a long time to converge to $f$? Consider:

1. Two cliques not "well connected" vs.

2. Completely connected.

(1) should take longer than (2).

In e.g. (1), let $f = 1_S$, where $S$ is one of the cliques. $E(f)$ is small.

Roughly: fast convergence $\Rightarrow$ high $E(f)$
slow convergence $\Rightarrow$ low $E(f)$

**Global Variance & Global Mean**

Let $u \in \Pi$, and $f : V \rightarrow \mathbb{R}$

- $u$ randomly selected from $\Pi$

- $f(u)$ is a random variable $u$ with the following parameters.
  - mean: $E[f] = E[f]
  - variance: $Var(f) = \mathbb{E}_{u \in \Pi} [f(u) - E[f]]^2
  = \mathbb{E}_{u \in \Pi} [f^2(u) - E[f]^2
  = \frac{1}{2} \mathbb{E}_{u,v \in \Pi} [(f(u) - f(v))^2]

**Known as the global variance**. It's taken into account all $u, v$ pairs, not just those with edges but in the clique.

**Spectral graph theory** compares local and global variances.

E.g. if for all $f$, $E(f)$ large or $Var(f)$, graph is an expander.
Inner Products

Want a way to measure the "similarity" of functions on a graph.

Let \( f, g : V \to \mathbb{R} \).

\[
\langle f, g \rangle_{\pi} = \sum_{u \in V} f(u)g(u)
\]

measures similarity.

similar to an inner product, but scaled by \( \pi \).

\[
\langle f, g \rangle = \begin{bmatrix} f(u_1) \\
                            f(u_2) \\
                            \vdots \\
                            f(u_n) \end{bmatrix}, \quad \begin{bmatrix} g(u_1) \\
                                                         g(u_2) \\
                                                         \vdots \\
                                                         g(u_n) \end{bmatrix}
\]

\[
\langle f, g \rangle_{\pi} = f(u_1)g(u_1) + f(u_2)g(u_2) + \cdots + f(u_n)g(u_n)
\]

Note: \( \langle f, g \rangle_{\pi} = \langle g, f \rangle_{\pi} \) (obvious) associativity

\( \langle a \cdot f + g, h \rangle_{\pi} = a \langle f, h \rangle_{\pi} + \langle g, h \rangle_{\pi} \) linearity

\( \langle f, f \rangle_{\pi} \geq 0 \), \( \langle f, f \rangle_{\pi} = 0 \) if \( f \equiv 0 \)