**Max Cut**

\[ G = (V, E) \], find \( S \subseteq V \) such that the cardinality of the cut-set is maximized. i.e. \( C = \{ (u, v) \in E : u \in S, v \in V \setminus S \} \)

\[ \max |C| \]

\[ \text{not max, } \quad \text{max, } \]

**Theorem:** (the decision version of) max-cut is NP-complete [Karp, 1972].

**ILP formulation:** Let's have a variable \( x_{uv} \) for each edge \( (u, v) \) and a variable \( y_u \) for each vertex \( u \in V \).

**ILP**

\[
\begin{align*}
\max & \quad \sum_{(u,v) \in E} x_{uv} \\
\text{s.t.} & \quad x_{uv} \leq y_u + y_v \quad \forall (u,v) \in E \\
& \quad x_{uv} \leq 2 - (y_u + y_v) \\
& \quad x_{uv} \in \{0,1\}
\end{align*}
\]

**LP**

\[
\begin{align*}
\text{relaxation} & \quad 0 \leq x, y \leq 1
\end{align*}
\]

If \( y_u = y_v \), then \( x_{uv} = 0 \). If \( y_u \neq y_v \), then \( x_{uv} = 1 \).

So \( x_{uv} \) is an indicator variable for edges in the cut-set for the ILP formulation.
A few observations:

1. Since this is a maximization problem, so \( LP > 1LP = OPT \)
2. If \( y_v = \frac{1}{2} \) \( \forall v \in V \), then \( x_{uv} = 1 \) \( \forall (uv) \in E \). So this is the LP optimal solution! So \( LP = 1E1 \)
3. With a naive, greedy algorithm, we have \( \text{max-cut} \geq \frac{1}{2} |E| \). So \( |E| \geq |E| \)

Putting these together, we have: \( \text{OPT} \leq LP \leq 2 \cdot \text{OPT} \)

**Rounding of LP solution**

Randomized rounding scheme: for each \( y_v \), put \( v \in S \) with probability \( y_v \). Then the expected \# edges cut is:

\[
\sum_{(uv) \in E} \Pr[(u,v) \text{ in cut}] = \sum_{(uv) \in E} y_u (1-y_v) + y_v (1-y_u).
\]

Now, for all \( y_u, y_v \in [0, 1] \), we have

\[
y_u (1-y_v) + y_v (1-y_u) \geq \frac{1}{2} \min \{ y_u + y_v, (1-y_u) + (1-y_v) \} = \frac{1}{2} x_{uv}
\]

so we have \( \mathbb{E}[\text{rounding}] \geq \frac{1}{2} LP \geq \frac{1}{2} \text{OPT} \)

\( \Rightarrow \) \( \frac{1}{2} \)-approximation for max-cut!

Turns out all LP relaxations + rounding schemes are \( \frac{1}{2} \)-approximation at best, so LP is pretty bad. Need to try something else. Also the optimal solution for the LP relaxation is trivial, so that's pretty stupid. So we need to try something else.
Quadratic Unconstrained Binary Optimization (QUBO).

We present a different formulation of the Max-cut problem:

\[
\text{QUBO} \quad \max \frac{1}{2} \sum_{u \neq v} (1 - x_u x_v) \quad \longrightarrow \quad \min \ x^T Q x
\]

\[
\text{s.t. } x_u \in \{-1, 1\} \quad \forall u \in V.
\]

We want to find an SDP relaxation for QUBO.

**#1 Rank relaxation**

\[
x^T Q x = \text{Tr}(x^T Q x) = \text{Tr}(Q x x^T) = \text{Tr}(Q X)
\]

For the QUBO problem, \( X \) has properties:

\[
X \succeq 0 \quad X_{ii} = x_i^2 = 1 \quad \text{and} \quad \text{rank}(X) = 1.
\]

So

\[
\min \ \text{Tr} Q X
\]

\[
\text{s.t. } X_{ii} = 1 \quad \text{rank relaxation}
\]

\[
X \succeq 0 \quad \text{rank}(X) = 1
\]

**SDP formulation**

* Allowing \( \text{rank}(X) \geq 1 \) is the relaxation step, that produces the SDP formulation. This is known as "lifting", as though into a higher dimension.
#2 Lagrangian duality

\[
\begin{align*}
\min & \quad x^T Q x \\
\text{s.t.} & \quad x_i^2 - 1 = 0
\end{align*}
\]

we find the lagrangian:

\[
L(x, \lambda) = x^T Q x - \sum_{i=1}^{n} \lambda_i (x_i^2 - 1)
\]

\[
= x^T (Q - \Lambda) x + \text{Tr} \Lambda
\]

Then the dual function is \( g(\lambda) = \inf_x L(x, \lambda) \). For this to be bounded below, we need the implicit constraint that \( Q - \Lambda \succeq 0 \)

Then \( \inf_x L(x, \lambda) = \text{Tr} \Lambda \), so the dual problem is

\[
\begin{align*}
\max & \quad g(\lambda) = \text{max} \text{Tr} \Lambda \\
\text{s.t.} & \quad Q - \Lambda \succeq 0 \\
& \quad \Lambda \text{ diagonal}
\end{align*}
\]

This is an SDP! We added the constraint that \( Q - \Lambda \succeq 0 \).

Note that if we take the dual of this SDP, we obtain the primal version in (#1 Rank relaxation)

\[
\begin{align*}
\min & \quad x^T Q x \\
\text{s.t.} & \quad x_i^2 = 1 \\
& \quad (Q \text{ QUBO})
\end{align*}
\]

**Primal-Dual Pair of SDPs**

\[
\begin{align*}
\text{(P)} & \quad \min \text{Tr} \Lambda \\
\text{X} & \quad \Lambda \text{ diagonal}
\end{align*}
\]

**Lagrangian Duality**

\[
\begin{align*}
\text{Rank Relax}
\end{align*}
\]
Now we have the SDP formulation, for which we can find the optimum solution \( X \). How do we recover the original assignment for max cut? Goemans and Williamson tell us a way to do this.

1. Factorize \( X = V^T V \), \( V \in \mathbb{R}^{r \times n} \), not necessarily rank 1!

\[
V = \begin{bmatrix} v_1 & \ldots & v_i & \ldots & v_n \end{bmatrix}^T, \text{ each } X_{ij} = v_i^T v_j
\]

2. Assign each \( v_i \) to a point on the unit sphere in \( \mathbb{R}^r \)

3. Choose a random hyperplane, and assign each \( x_i^* \) to be +1 or -1 depending on which side of the hyperplane \( v_i \) lies on. (Fig 1).

What is the expected value of this rounding scheme? We have

\[
ALG = \mathbb{E} \left( \frac{1}{2} \sum (1 - x_i^* x_j^*) \right) = \frac{1}{2} \sum \mathbb{E} (1 - x_i^* x_j^*) = \frac{1}{2} \sum 2 \times \Pr \left( v_i, v_j \text{ are on different sides} \right) = \frac{1}{2} \sum \frac{2}{\pi} \Theta_{ij}
\]

Also, \( SDP = \frac{1}{2} \sum (1 - X_{ij}) \). We want to find \( d \) such that: \( d_{SDP} \leq ALG \leq SDP \).

\[
d \left( \frac{1}{2} \sum (1 - X_{ij}) \right) \leq \frac{1}{2} \cdot \frac{2}{\pi} \sum \Theta_{ij}
\]

\[
d (1 - X_{ij}) \leq \frac{2}{\pi} \Theta_{ij} = \frac{2}{\pi} \arccos (v_i^T v_j) = \frac{2}{\pi} \arccos (X_{ij})
\]

\( d = 0.878 \) will satisfy this inequality for all \( X_{ij} \in [0, 1] \).
So for $\alpha = 0.878$ we have $\alpha_{SDP} \leq \alpha_{ALG} \leq \alpha_{OPT} \leq \alpha_{SDP}$.

Finally, to find the bounds on $OPT$ we observe that:

$OPT \leq SDP$ (relaxation)

$OPT \geq ALG$ (since $OPT$ is max).

Putting together these inequalities we have:

$\alpha_{SDP} \leq \alpha_{ALG} \leq \alpha_{OPT} \leq \alpha_{SDP}$

$\alpha_{OPT} \leq \alpha_{ALG} \leq \alpha_{OPT}$

so we have found a $0.878$-approximation for max-cut! \[ \]

Other results

Def: An algorithm is an $(a, b)$-approximation if, if $OPT = a$, then $ALG = b$.

Ex: Goemans Williamson is a $(\frac{6}{2}, 0.878)$-approximation for max-cut, as we have proved above.

Now if we normalize max-cut (such that $OPT = 1$ means all edges are cut), then we can prove that GW is also a $(1-\epsilon, 1-\Omega(\sqrt{\epsilon}))$-approximation.
Proof: Given SDP solution $X = V^TV$, we have our SDP optimum at:

$$\text{SDP} = \sum_{(i,j) \in E} \frac{1 - v_i^T v_j}{2|E|} \Rightarrow \text{OPT} = 1 - \varepsilon, \quad \varepsilon > 0$$

Note that this SDP is normalized with factor $\frac{1}{|E|}$. Now we can define $\varepsilon_{ij}$:

$$\varepsilon_{ij} : 1 - \varepsilon_{ij} = \frac{1 - v_i^T v_j}{2}$$

Then we have $\text{SDP} = \sum_{(i,j) \in E} \frac{1 - \varepsilon_{ij}}{|E|} = 1 - \sum_{(i,j) \in E} \frac{\varepsilon_{ij}}{|E|} \Rightarrow 1 - \varepsilon$.

Now we can follow the proof for (GW) rounding to find $E(\text{rounding})$:

$$E(\text{rounding}) = \text{ALG} = \frac{1}{|E|} \sum_{(i,j) \in E} \Pr \left[ v_i^*, v_j^* \text{ are on different sides} \right]$$

$$= \frac{1}{|E|} \sum_{(i,j) \in E} \frac{\arccos(v_i^T v_j)}{\pi}$$

$$= \frac{1}{|E|} \sum_{(i,j) \in E} \frac{\arccos(2 \varepsilon_{ij} - 1)}{\pi}$$

Using a lemma from Taylor, we see that:

$$\cos(\pi - \delta) \approx -1 + \delta^2$$

$$\frac{\arccos(\delta^2 - 1)}{\pi} \approx 1 - \frac{\delta}{\pi} \Rightarrow 1 - \Omega(\delta)$$

So

$$\text{ALG} \geq \frac{1}{|E|} \sum_{(i,j) \in E} \left[ 1 - \Omega(\sqrt{\varepsilon_{ij}}) \right]$$

$$= 1 - \frac{1}{|E|} \sum_{(i,j) \in E} \Omega(\sqrt{\varepsilon_{ij}})$$

and by the Minkowski inequality (?), this is upper bounded by

$$\geq 1 - \Omega \left( \sqrt{\sum_{(i,j) \in E} \varepsilon_{ij}} \right)$$

$$\geq 1 - \Omega(\sqrt{\varepsilon})$$