LP Relaxations & Approximation Algorithms

Vertex Cover: Input: undirected graph \( G = (V, E) \)

Goal: Find \( S \subseteq V \) so that 1) \( \forall \{i,j\} \in E, \{i,j\} \cap S \neq \emptyset \), 2) \( |S| \) minimized

NP-Hard to find the minimum VC of a graph.

Goal: Approximation — find a solution that is "comparable" with the minimum VC.

\( \alpha \)-approximation. An algorithm is \( \alpha \)-approximation for VC (or any other minimization problem)

if it always outputs a solution with objective value \( ALG \leq \alpha \cdot OPT \) \((\alpha \geq 1)\).

Integer Linear Program for Vertex Cover.

Let \( x_i = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{if } i \notin S \end{cases} \)

ILP: minimize \( \sum_{i \in V} x_i \)

st. \( x_i \in \{0,1\} \quad \forall i \in V \)

\( x_i + x_j \geq 1 \quad \forall \{i,j\} \in E \)

Still NP-Hard to solve the ILP — because of the integral constraint \( x_i \in \{0,1\} \)

Relaxation: \( x_i \in [0,1] \rightarrow \text{relax} \)

LP relaxation for Vertex Cover.

LP: minimize \( \sum_{i \in V} x_i \)

st. \( 0 \leq x_i \leq 1 \quad \forall i \in V \)

\( x_i + x_j \geq 1 \quad \forall \{i,j\} \in E \)

- Efficient (poly-time) Solvable!

- Every solution for ILP remains valid in LP (because of relaxation): LP \( \leq \) ILP

- However, there might be new fractional solutions.

1

1

integral
solution for ILP

new fractional solutions.

1

.7

.3

\( \frac{1}{2} \)

\( \frac{1}{2} \)

\( \frac{1}{2} \)

: optimal solution

for LP = 1.5

what does it mean

in Vertex Cover?

Rounding procedure. Take a fractional solution, and convert it to an integral solution.

(But may lose some value in objective function).
Rounding for Vertex Cover. Let \( x^*_i = \begin{cases} 1 & \text{if } x_i > \frac{1}{2} \\ 0 & \text{if } x_i < \frac{1}{2} \end{cases} \)

Claim: \( x^*_i \) is valid soln. for ILP. \( x^*_i \in \mathbb{R} \)

Proof: 1) \( x^*_i \in \{0,1\} \forall i \in V \); 2) \( x_i + x_j^* \geq 1 \iff \text{at least one of } x_i, x_j^* \geq \frac{1}{2} \)

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1 \quad \text{rounding} \quad 1 \\
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What about objective value?

Claim: Rounding \( \leq 2 \sum_{i \in S} x_i^* \leq 2 \sum_{i \in S} x_i \)

Proof. Because \( x_i^* \leq 2x_i \forall i \in S \)

Claim. Solving LP relaxation + Rounding is 2-approx for Vertex Cover.

Proof. Rounding \( \leq 2 \cdot LP \leq 2 \cdot ILP \)

Integrality Gaps. We proved that \( LP \in [\frac{1}{2} ILP, \frac{3}{2} ILP] \) (to see the lower bound: \( ILP \leq \text{rounding} \leq 2 \cdot LP \)).

- \( LP \) is poly-time solvable. \( ILP \) is NP-Hard. \( LP \) is an "estimator" of \( ILP \).
- How good is this estimator? \( \frac{1}{2} \) is a factor of 2.
- Can we improve the analysis to make it better? \( \frac{1}{2} \) can't be better than \( \frac{4}{3} \).

- Consider \( K_n \)-complete graph with \( n \) vertices.

\[ ILP \ (\min VC) = n-1 \]

\[ LP \leq n/2 \ (x_i = 1/2 \forall i \text{ is a feasible soln.}) \]

\[ ILP/LP \geq 2 - O(\frac{1}{n}) \]. — an example where \( LP \) is indeed off by factor of 2.

Integrality gap instance is a certificate.

We say \( K_n \) is a 2-integrality gap instance for \( LP \). — A certificate on the bad estimation of \( LP \).
Set Cover
Input: Universe $U = \{1, 2, ..., n\}$ and $S_1, S_2, ..., S_m \subseteq U$

Goal: Find smallest $I \subseteq \{1, 2, ..., m\}$, s.t. $\bigcup_{i \in I} S_i = U$

Remark: Vertex Cover is a special case of Set Cover. (Universe is the set of edges, each set corresponds to the set of incident edges of a vertex)

ILP for set cover:

Let $x_i = \begin{cases} 1 & \text{if } i \in I \\ 0 & \text{otherwise} \end{cases}$

Minimize $\sum_{i=1}^{m} x_i$

St. $x_i \in \{0, 1\}$ \quad $\forall i \in \{1, 2, ..., m\}$

$\sum_{i:u \in S_i} x_i \geq 1$ \quad $\forall u \in U$

LP relaxation:

Minimize $\sum_{i=1}^{m} x_i$

St. $x_i \in \mathbb{R}$ \quad $\forall i \in \{1, 2, ..., m\}$

$\sum_{i:u \in S_i} x_i \geq 1$ \quad $\forall u \in U$

Claim: $\text{ILP} \geq \text{LP}$ (because of relaxation).

Rounding. Idea: given LP solution $[x_i]$, treat $x_i$ as the probability that $i \in I$

Alg RandomPick:
- For each $i \in [m]$, let $i \in I$ w.p. $x_i$ independently.
- Return $I$.

Claim: $\mathbb{E}|I| = \sum_{i=1}^{m} x_i$

Claim: For each $u \in U$, $\mathbb{P}[u \in \bigcup_{i \in I} S_i] = 1 - \frac{1}{e}$

Proof.

$\mathbb{P}[u \in \bigcup_{i \in I} S_i] = \sum_{i:u \in S_i} \mathbb{P}[i \in I] = \sum_{i:u \in S_i} (1 - x_i)$

$\leq \exp\left(-\sum_{i:u \in S_i} x_i\right)$ (we use the fact $1 - t \leq e^{-t}$ for $t \geq 0$)

$= \exp\left(-\sum_{i:u \in S_i} x_i\right)$

$\leq \exp\left(-1\right)$ (because of the LP constraint $\sum_{i:u \in S_i} x_i \geq 1$)

Remark. RandomPick returns a set $I$ that covers each element w.p. $\geq 1/e$.
Repeat a few times so that each element covered with higher prob.

Alg RandomizedRound:
- Iterate $[2\ln n]$ times, at iteration $j$, let $I_j \leftarrow \text{RandomPick}$
- Return $I = \bigcup_{j=1}^{[2\ln n]} I_j$.
Claim \( E[|I|] \leq \sum_j E[I_j] \leq \left( \frac{4\ln n}{\varepsilon^2} \right) \cdot \text{OPT} \).

Claim For each \( u \in U \), \( Pr[u \in \cup_{i \in I_j} S_i] \geq 1 - \left( \frac{1}{\varepsilon} \right)^{\left( \frac{4\ln n}{\varepsilon^2} \right)} \cdot \text{OPT} \)
\[ \geq 1 - \left( \frac{1}{\varepsilon} \right)^{\left( \frac{4\ln n}{\varepsilon^2} \right)} \approx 1 - \frac{1}{\varepsilon^2} \]

Corollary \( Pr[I \text{ covers } U] = 1 - Pr[\exists u : u \notin \cup_{i \in I_j} S_i] \)
\[ \geq 1 - \sum_{u \in U} Pr[u \notin \cup_{i \in I_j} S_i] \quad \text{(union bound)} \]
\[ \geq 1 - \frac{n}{\varepsilon^2} = 1 - \frac{1}{\varepsilon}. \]

Theorem W.p. \( \frac{1}{2} - \frac{1}{n} \geq 0.4 \quad \text{(for large enough } n \text{), RandomizedRound returns } I \)
\( \text{st} \quad |I| \leq \text{OPT}(4\ln n + O(1)) \), \( I \) is a set cover for \( U \).

Proof 1) \( Pr[|I| \leq \text{OPT}(4\ln n + O(1))] = 1 - Pr[|I| > \text{OPT}(4\ln n + O(1))] \)
\[ \geq 1 - \left( \frac{4\ln n + O(1)}{\varepsilon^2} \right) \cdot \text{OPT} \quad \text{(Markov Ineq.)} \]
\[ \geq 1 - \frac{1}{2} = \frac{1}{2} \]

2) \( Pr[I \text{ covers } U] \geq 1 - \frac{1}{n} \text{ by the corollary.} \)

Therefore, \( Pr[|I| \leq \text{OPT}(4\ln n + 2) \text{ and } I \text{ covers } U] \)
\[ \geq 1 - (1 - \frac{1}{2}) - \left( 1 - (1 - \frac{1}{n}) \right) = \frac{1}{2} - \frac{1}{n} \quad \text{(union bound).} \]