Probability (2)

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Review

• A finite probability space is denoted by \((S, P)\) where
  • \(S\) is a finite set (the sample space), and
  • \(P\) is a function \(S \to [0,1]\) (the probability measure) such that
    \[
    \sum_{x \in S} P(x) = 1
    \]

Whenever hearing “probability”, make sure that you are clear what the probability space is: what is the sample space and what is the probability measure on it.
Conditional Probability

If $B \neq \emptyset$, the probability of event $A$ conditioned on the fact that $B$ happens, $P(A|B) = \frac{P(A \cap B)}{P(B)}$. 

\[ P(A|B) = P(A \cap B) P(B) \]
Independent Events

Events A and B are independent if

\[ P(A|B) = P(A). \]

An equivalently definition of independent events A and B:

\[ P(A \cap B) = P(A) \cdot P(B). \]

tossing a fair coin 1 time

\[ A = \{ \text{the sum of the two outcomes is 0} \} \]
\[ B = \{ \text{1} \} \]
• Drawing \textit{(with replacement)} two cards from a standard deck. Let
\[ E = \{ \text{the first card is a King} \} \]
\[ F = \{ \text{the second card is a King} \} \]

\[ P(E \cap F) = \frac{4}{52} \cdot \frac{4}{52} \]

\[ P(E) = P(F) = \frac{4}{52} \] \quad (E \text{ and } F \text{ are independent})
• Drawing (without replacement) two cards from a standard deck. Let

\[ E = \{ \text{the first card is a King} \} \]

\[ F = \{ \text{the second card is a King} \} \]

\[
P(E) = \frac{4}{52}
\]

\[
P(F) = \begin{cases} 
\frac{3}{51}, & \text{if } E \text{ occurs} \\
\frac{4}{51}, & \text{if } E \text{ doesn't occur.}
\end{cases}
\]

\[
P(E \cap F) = P(F|E) \cdot P(E) = \frac{4}{52} \times \frac{3}{51} = \frac{1}{221} \neq P(E) \cdot P(F)
\]
Bayes’s Formula (1)

For any two events $A$ and $B$,

$$P(A) = P(A|B) \cdot P(B) + P(A|\overline{B}) \cdot P(\overline{B})$$
Bayes’s Formula (2)

For any two events \( A \) and \( B \),

\[
P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B|A) \cdot P(A) + P(B|\overline{A}) \cdot P(\overline{A})}
\]
Exercise 1

- Two urns:
  - Urn #1 has 10 gold coins and 5 silver coins
  - Urn #2 has 2 gold coins and 8 silver coins
First randomly pick an urn then randomly pick a coin from the urn. What is the probability it is a gold coin?

\[ P(G \mid B) = \frac{10}{10+5} = \frac{2}{3} \]
\[ P(B) = \frac{1}{2} \quad P(\overline{B}) = \frac{1}{2} \]
\[ P(G \mid \overline{B}) = \frac{2}{2+8} = \frac{1}{5} \]

\[ P(G) = P(G \mid B) \cdot P(B) + P(G \mid \overline{B}) \cdot P(\overline{B}) \]
\[ = \frac{2}{3} \cdot \frac{1}{2} + \frac{1}{5} \cdot \frac{1}{2} = \frac{1}{3} + \frac{1}{10} = \frac{13}{30} \]
Exercise 2

• Two urns:
  Urn #1 has 10 gold coins and 5 silver coins
  Urn #2 has 2 gold coins and 8 silver coins
First randomly pick an urn then randomly pick a coin from the urn. It turns out that the coin is golden. What is the probability that urn #1 was picked?

\[ P(B\mid G) = \frac{P(G\mid B)P(B)}{P(G\mid B)P(B) + P(G\mid \bar{B})P(\bar{B})} \]

\[ = \frac{2/3 \cdot 1/2}{2/3 \cdot 1/2 + 1/2 \cdot 5/13} = \frac{1/3}{13/30} = \frac{10}{13} \]
Random Variable

A random variable is a function $X: S \rightarrow \mathbb{R}$ (from the sample space to the reals)

- Coin-tossing:
  - $S = \{H, T\}$
  - $X(H) = 1$
  - $X(T) = 0$

- Shooting game:
  - $X(V) = V$, $V \in [1, 10]$
Expectation of a Random Variable

• The expected value of a random variable $X$, denoted by $\mathbb{E}[X]$, is defined as

$$\mathbb{E}[X] = \sum_{s \in S} P(s)X(s)$$

• Example - Fair coin tossing: define $X(H)=1$, $X(T)=0$.

$$\mathbb{E}(X) = P(H) \cdot X(H) + P(T) \cdot X(T)$$

$$= \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 0 = \frac{1}{2}$$
Shooting competition

<table>
<thead>
<tr>
<th></th>
<th>10</th>
<th>9</th>
<th>8</th>
<th>7</th>
<th>6</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>James</td>
<td>45%</td>
<td>30%</td>
<td>27%</td>
<td>13%</td>
<td>1%</td>
<td>0</td>
</tr>
<tr>
<td>Venny</td>
<td>55%</td>
<td>18%</td>
<td>17%</td>
<td>5%</td>
<td>3%</td>
<td>2%</td>
</tr>
</tbody>
</table>

Who is more likely to win in a competition?

James: $\mathbb{E}(X) = 45\% \cdot 10 + 30\% \cdot 9 + 27\% \cdot 8 + 13\% \cdot 7 + 1\% \cdot 6$

$= 4.5 + 2.7 + 2.16 + 0.91 + 0.06$

Venny: $\mathbb{E}(X) = 55\% \cdot 10 + 18\% \cdot 9 + 17\% \cdot 8 + 5\% \cdot 7 + 3\% \cdot 6$

$= 5.5 + 1.62 + 1.36 + 0.35 + 0.18 + 0.1$
More examples

• What is the expected outcome of rolling a dice?

$$E(x) = \frac{1}{6} \cdot 1 + \frac{1}{6} \cdot 2 + \frac{1}{6} \cdot 3 + \frac{1}{6} \cdot 4 + \frac{1}{6} \cdot 5 + \frac{1}{6} \cdot 6$$

$$= \frac{1}{6} (1+6) \frac{6}{2} = \frac{1}{6} \cdot 21 = \frac{7}{2}$$
Rolling a fair dice, what is the expectation of the square of the outcomes?

$$E(X^2) = \sum_{s \in S} P(s) \cdot X^2(s) = \frac{1}{6} \sum_{s \in S} X^2(s)$$

$$= \frac{1}{6} \left( 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 \right) = 9 1/6$$
More examples

• How about rolling a dice twice? What is the expectation of the sum of the two outcomes?

<table>
<thead>
<tr>
<th>X($)</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>P($)</td>
<td>1/36</td>
<td>1/18</td>
<td>1/12</td>
<td>1/9</td>
<td>5/36</td>
<td>1/6</td>
<td>1/9</td>
<td>1/12</td>
<td>1/18</td>
<td>1/36</td>
<td>1/36</td>
</tr>
</tbody>
</table>

$$E(X) = 2 \times \frac{1}{36} + 3 \times \frac{1}{18} + 4 \times \frac{1}{12} + 5 \times \frac{1}{9} + 6 \times \frac{5}{36} + 7 \times \frac{1}{6} + 8 \times \frac{1}{9} + 9 \times \frac{1}{12} + 10 \times \frac{1}{18} + 11 \times \frac{1}{36} + 12 \times \frac{1}{36} = 7.$$
Linearity of Expectation

• For random variables $X$ and $Y$ (which may be dependent),

$$E[X + Y] = E[X] + E[Y]$$

• More generally, for random variables $X_1, X_2, ..., X_n$ and constants $c_1, c_2, ..., c_n$,

$$E[c_1X_1 + \cdots + c_nX_n] = c_1E[X_1] + \cdots + c_nE[X_n]$$
Better way

• Expected outcome of rolling a dice *twice*?

  What is the expectation of the sum of the two outcomes.

  $X_1$: the random variable of the first roll.

  $X_2$: second roll

  $\mathbb{E}[X_1 + X_2] = \mathbb{E}[X_1] + \mathbb{E}[X_2] = 7/2 + 7/2 = 7$
Exchanging Gifts

- At a Christmas party, $n$ friends each bought a gift box and mixed them together. Later, each person randomly draws a gift box from the pile. On average, how many people will get back their own gift?

If $n=3$, $3! = 6$

\[
\begin{align*}
1 & \quad 2 & \quad 3 \\
1 & \quad (3 & \quad 2) \\
(2 & \quad 3) & \quad 1 & \quad (3 & \quad 1) \\
(3 & \quad 1) & \quad 2 & \quad (1 & \quad 3) \\
1 & \quad 2 & \quad 3 & \quad 1 & \quad 2 & \quad 3
\end{align*}
\]

\[
3 \times \frac{1}{6} + 1 \times \frac{1}{6} + 1 \times \frac{1}{6} + 0 \times \frac{1}{6} + 0 \times \frac{1}{6} = \frac{1}{2} + \frac{1}{2} = 1
\]
Exchanging Gifts

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$$X_i = \begin{cases} 1, & \text{the } i^{th} \text{ person gets back his/her own gift} \\ 0, & \text{otherwise} \end{cases}$$

$$E(X_i) = 1 \times \frac{1}{n} + 0 \times \frac{n-1}{n} = \frac{1}{n}$$

Binary random variable like $X_i$ is called an indicator random variable.

$$E \left[ \sum_{i=1}^{n} X_i \right] = \sum_{i=1}^{n} E(X_i) = \sum_{i=1}^{n} \frac{1}{n} = 1$$