Probability (1)

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Definition

- A finite **probability space** is denoted by $(S, P)$ where
  - $S$ is a finite set (the *sample space*), and
  - $P$ is a function $S \rightarrow [0,1]$ (the *probability measure*) such that

\[
\sum_{x \in S} P(x) = 1
\]

Whenever hearing “probability”, make sure that you are clear what the probability space is: *what is the sample space and what is the probability measure on it*. 
Probability Space \((S, P)\)

- An **outcome** is a point in \(S\).
- An **event** is a subset of \(S\).

Throw a dice

\[
S = \{1, 2, 3, 4, 5, 6\}
\]

\[
E_1 = \{ \text{all the outcomes less than 4} \} = \{ 1, 2, 3 \}
\]

\[
E_2 = \{ 1 \}
\]
Uniform Distribution

- Every point in $S$ is *equiprobable*.

\[ P(a) = \frac{1}{|S|} \]

E.g., fair dice of 6 faces.

\[ P(X = 1) = \frac{1}{6} = P(X = 2) = P(X = 3) = \ldots \]
Probabilities of events

- Let $A$ be an event of probability space $(S, P)$. The probability of event $A$ is

$$P(A) := \sum_{a \in A} P(a).$$

Assume a fair dice.

$$P(E_1) = \sum_{a \in \{1, 2, 3\}} P(a) = \frac{1}{6} \cdot 1 + \frac{1}{6} \cdot 1 + \frac{1}{6} \cdot 1 = \frac{1}{2}$$
Examples

• Toss a fair dice twice, what is the probability that the two outcomes add up to 5?

$$S = \{(1,1), (1,2), (2,1), (2,2), \ldots, (6,6)\}$$

$$|S| = 6 \times 6 = 36$$

$$E = \{(1,4), (2,3), (3,2), (4,1)\}$$

$$P(E) = \frac{4}{36} = \frac{1}{9}$$
Examples

• Toss a fair dice three times, what is the probability that the sum of the outcomes is less than 10?

\[ S = \{ (i, j, k) \mid 1 \leq i \leq 6, 1 \leq j \leq 6, 1 \leq k \leq 6 \} \]

\[ |S| = 6^3 \]

\[ E = \{ (i, j, k) \mid i + j + k < 10 \} = \{ (1,1,1), (1,1,2), \ldots, (1,1,6), (1,2,7), (1,2,1), \ldots, (1,2,6), (1,3,1), \ldots, (1,3,5) \} \]
\[(1, 4, 1), \ldots (1, 4, 4)\]
\[(1, 5, 1), \ldots (1, 5, 3)\]
\[(1, 6, 1), \ldots (1, 6, 2)\]

\[4\]
\[\#E_1 = (7 + 6 + \ldots + 2) + (6 + 5 + \ldots + 1) + (5 + 4 + \ldots + 1) + (4 + 3 + \ldots + 1) + (3 + 2 + 1) + (2 + 1)\]

\[= \frac{(2 + 6) \times 6}{2} + \frac{(6 + 1) \times 6}{2} + \frac{(5 + 1) \times 5}{2} + \frac{(4 + 1) \times 4}{2} + \frac{(3 + 1) \times 3}{2} + 3\]

\[= 28 + 21 + 15 + 10 + 6 + 3 = 82\]
\[ P(A) = \frac{|E|}{|S|} = \frac{8^2}{6^3} \]
Examples

• Drawing 5 cards from a standard deck of 52 poker cards (Four suits: clubs, spades, diamonds, hearts. Each suit has thirteen cards: A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K).
  - What is the probability of the five cards being Royal Flush (i.e., same-suit 10, J, Q, K, A)?

\[
\left| S \right| = \binom{52}{5} = \frac{52!}{(52-5)! \cdot 5!}
\]

\[
\frac{4}{\left| S \right|} = \frac{4}{\binom{52}{5}}
\]
Examples

• Drawing 5 cards from a standard deck of 52 poker cards (Four suits: clubs, spades, diamonds, hearts. Each suit has thirteen cards: A, 2, 3, ..., 10, J, Q, K).
  - What is the probability of the five cards being a **Straight Flush**? Straight flush is a poker hand containing five cards of sequential rank, all of the same suit, such as Q♥ J♥ 10♥ 9♥ 8♥ (a “queen-high straight flush”), but not a royal flush.

\[
\frac{(10 - 1) \times 4}{\binom{52}{5}} = \frac{36}{\binom{52}{5}}
\]
Examples

• Drawing 5 cards from a standard deck of 52 poker cards (Four suits: clubs, spades, diamonds, hearts. Each suit has thirteen cards: A, 2, 3, ..., 10, J, Q, K).
  - What is the probability of the five cards being a **Four of a Kind**? Four of a kind, also known as **quads**, is a poker hand containing four cards of the same rank and one card of another rank, e.g., $9\spadesuit 9\spadesuit 9\spadesuit 9\heartsuit J\diamond$ ("four of a kind, nines").

\[
\binom{3 \times 48}{C(\{52, 5\})}
\]
Examples

• Drawing 5 cards from a standard deck of 52 poker cards (Four suits: clubs, spades, diamonds, hearts. Each suit has thirteen cards: A, 2, 3, ..., 10, J, Q, K).
  - What is the probability of the five cards being a **Full House**? A full house is a poker hand containing three cards of one rank and two cards of another rank, such as 3♣ 3♣ 3♦ 6♣ 6♥.

\[
\frac{\binom{13}{1} \binom{4}{3} \binom{12}{2} \binom{4}{2}}{\binom{52}{5}}
\]
Examples

• Drawing 5 cards from a standard deck of 52 poker cards (Four suits: clubs, spades, diamonds, hearts. Each suit has thirteen cards: A, 2, 3, ..., 10, J, Q, K).

  - What is the probability of the five cards being a **Flush**? A flush is a poker hand containing five cards all of the same suit, but *not all of sequential rank*, such as K♣ 10♣ 7♣ 6♣ 4♣.

\[
\frac{\binom{13}{5} \times 4 - 10 \times 4}{\binom{52}{5}}
\]
Examples

• Drawing 5 cards from a standard deck of 52 poker cards (Four suits: clubs, spades, diamonds, hearts. Each suit has thirteen cards: A, 2, 3, ..., 10, J, Q, K).
  - What is the probability of the five cards being a Straight? A straight is a poker hand containing five cards of sequential rank, *not all of the same suit*, such as 7♠ 6♠ 5♠ 4♥ 3♥.

\[
\frac{(4^5 - 4) \times 10}{C(52, 5)}
\]