Dynamic External Hashing:
The Limit of Buffering

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(internal) Hashing!

One of the most important data structures in computer science!
External hashing!

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A cell has $b = 2$ words

Extremely useful in Database System!
Model and problem

Model: External memory model. Block size $b$ words, cache size $m$ words. Cost is “number of blocks read/write (I/Os)”

Problem: Maintain a hash table to support update and query.

Try to understand “the inherent tradeoff between queries and updates”

Hashing
Results
Previous results

- Hashing in the internal memory is well understood (under random inputs).

Knuth, 1970s: \( t_q = \frac{1}{2}(1 + 1/(1 - \alpha)) \), \( t_u = 1 + \frac{1}{2}(1 + 1/(1 - \alpha)^2) \). \( \alpha \): load factor: minimum storage should be use/storage actually used
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- In external memory (random inputs)

  Knuth, 1970s: Expected average cost of a query is \( 1 + 1/2^{\Omega(b)} \) I/Os, provided the load factor \( \alpha \) is less than a constant smaller than 1. Update has a similar bound.
Exact Numbers Calculated by D. E. Knuth

Table 2
AVERAGE ACCESES IN AN UNSUCCESSFUL SEARCH BY SEPARATE CHAINING

| Bucket size, b | 10%   | 20%   | 30%   | 40%   | 50%   | 60%   | 70%   | 80%   | 90%   | 95%   |
|               | 1.0048| 1.0187| 1.0408| 1.0703| 1.1065| 1.1488| 1.197 | 1.249 | 1.307 | 1.34  |
| 2             | 1.0012| 1.0088| 1.0269| 1.0581| 1.1036| 1.1638| 1.238 | 1.327 | 1.428 | 1.48  |
| 3             | 1.0003| 1.0038| 1.0162| 1.0433| 1.0898| 1.1588| 1.252 | 1.369 | 1.509 | 1.59  |
| 4             | 1.0001| 1.0016| 1.0095| 1.0314| 1.0751| 1.1476| 1.253 | 1.394 | 1.571 | 1.67  |
| 5             | 1.0000| 1.0007| 1.0056| 1.0225| 1.0619| 1.1346| 1.249 | 1.410 | 1.620 | 1.74  |
| 10            | 1.0000| 1.0000| 1.0004| 1.0041| 1.0222| 1.0773| 1.201 | 1.426 | 1.773 | 2.00  |
| 20            | 1.0000| 1.0000| 1.0000| 1.0000| 1.0028| 1.0234| 1.113 | 1.367 | 1.898 | 2.29  |
| 50            | 1.0000| 1.0000| 1.0000| 1.0000| 1.0000| 1.0000| 1.018 | 1.182 | 1.920 | 2.70  |

Table 3
AVERAGE ACCESES IN A SUCCESSFUL SEARCH BY SEPARATE CHAINING

| Bucket size, b | 10%   | 20%   | 30%   | 40%   | 50%   | 60%   | 70%   | 80%   | 90%   | 95%   |
|               | 1.0500| 1.1000| 1.1500| 1.2000| 1.2500| 1.3000| 1.350 | 1.400 | 1.450 | 1.48  |
| 2             | 1.0063| 1.0242| 1.0520| 1.0883| 1.1321| 1.1823| 1.238 | 1.299 | 1.364 | 1.40  |
| 3             | 1.0010| 1.0071| 1.0215| 1.0458| 1.0806| 1.1259| 1.181 | 1.246 | 1.319 | 1.36  |
| 4             | 1.0002| 1.0023| 1.0097| 1.0257| 1.0527| 1.0922| 1.145 | 1.211 | 1.290 | 1.33  |
| 5             | 1.0000| 1.0008| 1.0046| 1.0151| 1.0358| 1.0699| 1.119 | 1.186 | 1.268 | 1.32  |
| 10            | 1.0000| 1.0000| 1.0022| 1.0015| 1.0070| 1.0226| 1.056 | 1.115 | 1.206 | 1.27  |
| 20            | 1.0000| 1.0000| 1.0000| 1.0000| 1.0005| 1.0038| 1.018 | 1.059 | 1.150 | 1.22  |
| 50            | 1.0000| 1.0000| 1.0000| 1.0000| 1.0000| 1.0000| 1.001 | 1.015 | 1.083 | 1.16  |
Can we batch updates?

- However, in external memory model, disk reads/writes are expensive and powerful. Can we hope for lower than 1 I/O per update?
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- Maybe yes, if we have an $\Omega(b)$ main memory for buffering! Like numerous problems in external memory, e.g. stack. More: priority queue, buffer tree ...

Can the amortized update cost be something like $O(1/b^c)$ (for some $0 < c \leq 1$) for hashing?
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Can the amortized update cost be something like $O(1/b^c)$ (for some $0 < c \leq 1$) for hashing?

  The insertion cost must be $\Omega(1)$ I/Os if the query cost is required to be $O(1)$ I/Os.
Our results

Due to Knuth (on random inputs)

Insertion expected amortized

Due to Knuth (on random inputs)

1 + 1/2^{Ω(b)}

1 - O(1/b^{(c-1)/6})

Ω(1)

1 + Θ(1/b^c), c < 1

1 + Θ(1/b)

1 + Θ(1/b^c), c < 1

1 + Θ(1/b^c), c > 1

O(b^{c-1})

Ω(b^{c-1})

successful queries

upper bounds

lower bounds

expected average

Query
Our results

Due to Knuth (on random inputs)

\[ 1 + 1/2^{\Omega(b)} \]

\[ 1 - O(1/b^{(c-1)/6}) \]

\[ \Omega(1) \]

\[ O(1) \]

Insertion expected amortized

Due to Knuth (on random inputs)

\[ 1 + \Theta(1/b^c), \ c < 1 \]

\[ 1 + \Theta(1/b) \]

successful queries

Query expected average

upper bounds

lower bounds
Our results

Due to Knuth (on random inputs)

Almost a complete understanding for successful queries!
Other related results

- Upper bounds
  - Remove *ideal hash function* assumption [Carter and Wegman 1979], making query *worst-case* [i.e. Fredman, Komlos and Szemeredi, 1984] ... (internal)
  - Queries and updates in $1 + O(1/b^{1/2})$ I/Os with $\alpha = 1 - O(1/b^{1/2})$ [Jensen and Pagh, 2007]. (external, no memory)
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- Lower bounds (internal)
  Very sparse, only with some strong requirements, e.g., the algorithm is deterministic and query is worst-case [Dietzfelbinger et. al. 1994].
Other related results

- **Upper bounds**
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- **Lower bounds (internal)**
  - Very sparse, only with some strong requirements, e.g., the algorithm is *deterministic* and query is *worst-case* [Dietzfelbinger et. al. 1994].

- **Lower bounds in other dynamic external memory problems**
  - Only known are query-update tradeoffs for the *predecessor* [Fagerberg and Brodal 2003], *range reporting* [Yi 2009].
Technical details:
Lowerbounds
Preliminaries

- $U = \{0, 1, \ldots, u - 1\}$: universe. $|U| = u$.

- $m$: size of main memory. $n$: total number of items. $b$: size of one block. All in words.
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- Some mild assumptions
  - Atomic elements
  - $n \geq \Omega (m \log u \cdot b^{2c})$ for some constant $c > 0$
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- Some mild assumptions
  - Atomic elements
  - $n \geq \Omega(m \log u \cdot b^{2c})$ for some constant $c > 0$
  - Deterministic data structure + a random distrib. of inputs
    (Via a method similar to Yao's Minimax Principle) $\implies$ Randomized data structure
Observations

- Two extreme cases

- One extreme: only use a fixed mapping for all items.

\[ b = 2 \]

Update is expensive!
Observations

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  - Another extreme: for every $b$ items come, write to a new block.

$b = 2$

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Observations

- Two extreme cases
  - One extreme: only use a fixed mapping for all items.
    - Update is expensive!
  - Another extreme: for every $b$ items come, write to a new block.

- Also easy to see
  - If with only the information in memory, the hash table cannot locate the item, then querying it takes at least 2 I/Os.
The abstraction

Consider the layout of a hash table at any snapshot. Denote all the blocks on disk by $B_0, B_1, B_2, \ldots, B_d$ ($B_0 = M$). Let $f : U \rightarrow \{0, 1, \ldots, d\}$ be any function computable within memory.

When querying $x \in U$, $f(x)$: index of the first block the DS will probe. If $f(x) = 0$, the DS will still probe the memory.
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- We divide items inserted into 3 zones with respect to $f$.

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<th>Memory</th>
<th>Memory zone $M$: set of items stored in memory. $t_q = 0$.</th>
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<tbody>
<tr>
<td>Disk</td>
<td>Fast zone $F$: set of items $x$ such that $x \in B_{f(x)}$. $t_q = 1$.</td>
</tr>
<tr>
<td></td>
<td>Slow zone $S$: The rest of items. $t_q \geq 2$.</td>
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</tbody>
</table>
The key idea

The hash table can employ a family $\mathcal{F}$ of at most $2^m \log u$ distinct $f$’s.

Note that the current $f$ adopted by the hash table is dependent upon the already inserted items, but the family $\mathcal{F}$ has to be fixed beforehand.
Size of the slow zone is small.

- Suppose the hash table answers a successful query with an expected average cost of \( t_q = 1 + \delta \) I/Os. Consider the snapshot when \( k \) random items have been inserted.

\[
E[|S|] \leq m + \delta k.
\]

- Memory zone \( M: t_q = 0 \)
- Fast zone \( F: t_q = 1 \)
- Slow zone \( S: t_q \geq 2 \)

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  $$E[|S|] \leq m + \delta k.$$  

  Memory zone $M$: $t_q = 0$

  Fast zone $F$: $t_q = 1$

  Slow zone $S$: $t_q \geq 2$

  small!

- The high-probability version.

  **LEMMA 1.** Let $\phi \geq 1/b^{(c-1)/4}$ and let $k \geq \phi n$. At the snapshot when $k$ items have been inserted, with probability at least $1 - 2\phi$, $|S| \leq m + \frac{\delta}{\phi} k$. 
Basic idea of the lower bound proof

Consider any $f : U \to \{0, 1, \ldots, d\}$. For $i = 0, \ldots, d$, let $\alpha_i = |f^{-1}(i)|/u$, and we call $(\alpha_0, \alpha_1, \ldots, \alpha_d)$ the characteristic vector of $f$. 
Basic idea of the lower bound proof

Consider any \( f : U \rightarrow \{0, 1, \ldots, d\} \). For \( i = 0, \ldots, d \), let \( \alpha_i = |f^{-1}(i)|/u \), and we call \((\alpha_0, \alpha_1, \ldots, \alpha_d)\) the characteristic vector of \( f \).

After \( \phi n \) random insertions. Pick a fixed threshold \( \rho \). Assume \( \alpha_0 \geq \alpha_1 \geq \ldots \geq \alpha_d \)

\[
\begin{array}{cccc|ccc}
\alpha_0 & \alpha_1 & \alpha_2 & \ldots & \alpha_k & \ldots & \alpha_d \\
\end{array}
\]

(A) If \( \exists \) too many large \( \alpha_i \)'s, \( S \) too large, violating the query requirement (LEMMA 1).
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- After \( \phi n \) random insertions. Pick a fixed threshold \( \rho \). Assume \( \alpha_0 \geq \alpha_1 \geq \ldots \geq \alpha_d \)

(B) Else \( f \) is likely to distribute recent randomly inserted items evenly, leading to a high insertion cost.
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After \( \phi n \) random insertions. Pick a fixed threshold \( \rho \). Assume \( \alpha_0 \geq \alpha_1 \geq \ldots \geq \alpha_d \).

\( f \)

\[
\begin{array}{cccccccc}
\alpha_0 & \alpha_1 & \alpha_2 & \cdots & \alpha_j & \cdots & \alpha_k & \cdots & \alpha_d \\
\rho & & & & & \rho & & \\
\end{array}
\]

(A) If \( \exists \) too many large \( \alpha_i \)'s, (B) Else \( f \) is likely to distribute \( S \) too large, violating the query requirement (LEMMA 1).

Both hold with very high probability, even after taking union of all \( O(2^m \log u) \) different \( f \).
Upper bounds

Easy!

Logarithmic method

+ Query start from the last (biggest) layer
+ Tricks to keep the last layer large
Beyond hashing:
Subsequent and future work
Beyond hashing

Hashing (successful)
Beyond hashing

Membership

Problem: Maintain a set \( S \subseteq U \). Given an \( x \in U \), answer \( x \in S \)?

Again, tradeoffs between update and query

Membership: if \( t_q \leq 1 + \delta \) (\( 0 \leq \delta < 1/2 \)), then \( t_u \geq \Omega(1) \)

[Yi and Zhang 2009]

(1) Without atomic assumption
(2) Consider both successful and unsuccessful query

Hashing (successful)
Beyond hashing

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General Membership

General Hashing

Hashing (successful)
More problems

Lower bounds of other dynamic problems in the cell probe with cache setting.

1. predecessor, range-sum
2. union-find
3. . . .
The End

THANK YOU

Q and A

The Banff National Park