Redundant interdependencies boost the robustness of multilayer networks

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In the standard model of percolation on multilayer networks, a node is functioning only if its copies in all layers are simultaneously functioning. According to this model, a multilayer network becomes more and more fragile as the number of layers increases. In this respect, the addition of a new layer of interdependent nodes to a preexisting multilayer network will never improve its robustness. Whereas such a model seems appropriate to understand the effect of interdependencies in the simplest scenario of a network composed of only two layers, it may seem not reasonable for real systems where multiple network layers interact one with the other. It seems in fact unrealistic that a real system, such a living organism, evolved, through the development of multiple layers of interactions, towards a fragile structure. In this paper, we introduce a model of percolation where the condition that makes a node functional is that the node is functioning in at least two of the layers of the network. The model reduces to the standard percolation model for multilayer networks when the number of layers equals two. For larger number of layers, however, the model describes a scenario where the addition of new layers boosts the robustness of the system by creating redundant interdependencies among layers. We prove this fact thanks to the development of a message-passing theory able to characterize the model in both synthetic and real-world multilayer graphs.

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I. INTRODUCTION

Multilayer networks [1–3] are emerging as a powerful paradigm for describing complex systems characterized by the coexistence of different types of interactions. Multilayer networks represent an appropriate descriptive model for real networked systems in disparate contexts, such as social [4, 5], technological [6–8] and biological systems [9–11]. For example, global infrastructures are formed by several interdependent networks, such as power grids, water supply networks, and communication systems, and studying their properties require to account for the presence of such interdependencies [12]. Cell function and/or malfunction (yielding diseases) cannot be understood if the information on the different nature of the interactions forming the interactome (protein-protein interactions, signaling, regulation) are not integrated in a general multilayer scenario [9]. Similarly, the complexity of the brain is encoded in the different nature of the interactions existing at the functional and the structural levels [10, 11].

A multilayer networks is composed of a set of networks forming its layers [1–3]. Nodes can be connected within and across layers. It has been shown that multilayer networks are much more fragile than isolated networks just because of the presence of interdependencies among the layers of the system [12–18]. In particular, the fragility of the system increases as the number of layers increases [16, 19–22]. Such a feature has an intuitive explanation. In the standard percolation model for multilayer networks, the probability that a node is damaged equals to the probability that at least one of its interdependent nodes is damaged. As the number of layers increases, the probability of individual failures grows thus making the system more fragile. This scenario leads, however, to the conundrum: if the fragility of a system is increased by the number of layers of interactions, why are there so many real systems that display multiple layers of interactions? Further, the addition of new layers of interactions in a preexisting multilayer network has generally a cost, so it doesn't seem reasonable to spend resources just to make the system less robust. The purpose of the current paper is to provide a potential explanation by introducing a new model for percolation in networks composed of multiple interacting layers. In the model, we will assume that a node is damaged only if all its interdependent nodes are simultaneously damaged. The model is perfectly equivalent to the standard one when the number of layer equals two. Additional layers, however, provide the system with redundant interdependencies, generating backup mechanisms against the failure of the system, and thus making it more robust.

The robustness of multilayer networks in presence of redundant interdependencies is here investigated using a message-passing theory [13, 23–26] (also known as the cavity method). We build on recent advances obtained in standard interdependent percolation theory [19, 27–30] to propose a theory that is valid for multilayer networks with link overlap [5, 31] as long as the multilayer network is locally tree-like. This limitation is common to all message-passing approaches for studying critical phenomena on networks. Corrections have been recently proposed [32] on single networks to improve the performace of message-passing theory and similar approximations valid for loopy multilayer networks might be envisaged in the future.

II. REDUNDANT PERCOLATION MODEL ON MULTILAYER NETWORKS

We consider a multilayer network $\vec{G} = (G_1, G_2, \ldots, G_M)$ composed of M layers G_α with $\alpha = 1, 2, \ldots, M$. Every layer contains N nodes. Exactly one node with the same label appears in every individual layer. Nodes in the various layers sharing a common label are called *replica nodes*, and they are considered as interdependent on each other [33]. Nodes in the network are identified by a pair of labels (i, α), with $i = 1, 2, \ldots, N$ and $\alpha = 1, 2, \ldots, M$, the first one indicating the index of the node, and the second one standing for the index of the layer. For every node label i, the set of replica nodes is given by the M nodes corresponding to pairs of labels (i, α) with with $\alpha = 1, 2, \ldots, M$ (see Figure 1). When at least two replica nodes (i, α) and (i, α') are connected to two corresponding replica nodes (j, α) and (j, α') we say that the multilayer network displays link overlap.

Given a multilayer network as described above, we consider a percolation model where some of the nodes are initially damaged. We assume that the interdependencies are redundant, i.e., every node can be active only if at least one its interdependent nodes is also active. We refer to this model as "Redundant percolation model." As an order parameter for the model, we define the so-called Redundant Mutually Connected Giant Component (RMCGC). The nodes that belong to the RMCGC can be found by following the algorithm:

(i) The giant component of each layer α is determined, evaluating the effect of the damaged nodes in each single layer;

(ii) *Every replica node that has no other replica node in the giant component of its proper layer is removed from the network and considered as damaged*;

(iii) If no new damaged nodes are found at step (ii), then the algorithm stops, otherwise it proceeds, starting again from step (i).

The set of replica nodes that are not damaged when the algorithm stops belongs to the RMCGC.

The main difference with the standard percolation model [12] on multilayer networks and the consequent definition of Mutually Connected Giant Component (MCGC) is that step (ii) must be substituted with "Every replica node that has at least a single replica node not in the giant component of its proper layer is removed from the network and considered as damaged, i.e., if a replica node is damaged all its interdependent replica nodes are damaged" [12–16, 19–21]. In particular, the RMCGC and the MCGC are the same for $M = 2$ layers, but they differ as long as the number of layers $M > 2$. In the latter case, the RMCGC naturally introduces the notion of redundancy among interdependent nodes. As we will see in the following, the main effect of redundancy is to let the robustness of the system increases as the number of layers increases.

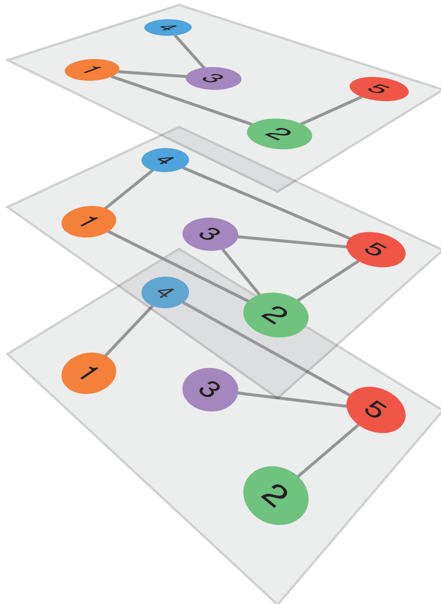

FIG. 1: A multilayer network with $M = 3$ layers, and $N = 5$ nodes is shown. Every node i has $M = 3$ interdependent replica nodes (i, α) with $\alpha = 1, 2, 3$. In this figure, triplets of replica nodes are also identified by their color.

III. MESSAGE-PASSING ALGORITHM

We assume that interactions within each layer α are described by elements $a_{ij}^{[\alpha]}$ of the adjacency matrix of the layer, indicating whether the replica nodes (i, α) and (j, α) are connected ($a_{ij}^{[\alpha]} = 1$) or not ($a_{ij}^{[\alpha]} = 0$) in layer α. Additionally, we consider a specific realization of the initial damage to the replica nodes indicated by the set $\{s_{i\alpha}\}$. The generic element $s_{i\alpha} = 0$ indicates that the replica node (i, α) has been initially damaged, whereas $s_{i\alpha} = 1$ indicates that the replica node (i, α) has not been initially damaged. Under these conditions, as long as the multilayer network is locally treelike, the following message-passing algorithm identifies the replica nodes that are in the RMCGC.

Each node i sends to a neighbor j a set of messages $n_{i\to j}^{[\alpha]}$ in every layer α where node i is connected to node j, i.e., with $a_{ij}^{[\alpha]} = 1$. These messages indicate whether ($n_{i\to j}^{[\alpha]} = 1$) or not ($n_{i\to j}^{[\alpha]} = 0$) node i connects node j to the RMCGC with links belonging to layer α. The message $n_{i\to j}^{[\alpha]} = 1$ if and only if all the following conditions are met:

(a) node i is connected to node j in layer α, and both nodes (i,α) and node (j,α) are not damaged, i.e., $s_{i\alpha}=s_{j\alpha}=a_{ij}^{[\alpha]}=1$;

(b) node i is connected to the RMCGC through at least one node $\ell\neq j$ in layer α;

(c) node i belongs to the RMCGC assuming that also node j belongs to the RMCGC. This conditions is satisfied if and only if, assuming that node j belongs to the RMCGC, node i is connected in at least two layers to the RMCGC.

If the previous conditions are not simultaneously met, then $n_{i\to j}^{[\alpha]}=0$. Put together, the former conditions lead to the algorithm for the messages $n_{i\to j}^{[\alpha]}$

$$n_{i\to j}^{[\alpha]}=\theta(v_{i\to j},2)\,a_{ij}^{[\alpha]}s_{j\alpha}s_{i\alpha}\left[1-\prod_{\ell\in N_\alpha(i)\setminus j}\left(1-n_{\ell\to i}^{[\alpha]}\right)\right].(1)$$

Here $N_\alpha(i)$ indicates the set of nodes that are neighbor of node i in layer α. The term $1-\prod_{\ell\in N_\alpha(i)\setminus j}\left(1-n_{\ell\to i}^{[\alpha]}\right)$ therefore will equal one if at least one message is arriving to node i from a neighboring node $\ell\neq j$, while it will be equal to zero, otherwise. $\theta(v_{i\to j},2)=1$ for $v_{i\to j}\geq 2$ and $\theta(v_{i\to j},2)=0$, otherwise. $v_{i\to j}$ indicates in how many layers node i is connected to the RMCGC assuming that node j also belongs to the RMCGC, i.e.,

$$\begin{aligned}v_{i\to j} &= \sum_{\alpha=1}^{M}\left[s_{i\alpha}\left(1-\prod_{\ell\in N_\alpha(i)\setminus j}\left(1-n_{\ell\to i}^{[\alpha]}\right)\right)\right.\\ &\left.+s_{i\alpha}s_{j\alpha}a_{ij}^{[\alpha]}\prod_{\ell\in N_\alpha(i)\setminus j}\left(1-n_{\ell\to i}^{[\alpha]}\right)\right].\end{aligned}\qquad(2)$$

Therefore $v_{i\to j}$ indicates the number of initially undamaged replica nodes (i,α) that either receive at least one positive messages from nodes $\ell\in N_\alpha(i)\setminus j$ or are connected to the undamaged replica nodes (j,α). Finally, the replica node (i,α) belongs to the RMCGC if (i) it is not damaged, (ii) it is connected to the RMCGC in layer α, and (iii) it receives at least another positive message in a layer $\alpha'\neq\alpha$. These conditions are summarized by

$$\begin{aligned}\sigma_{i\alpha} &= s_{i\alpha}\left(1-\prod_{\ell\in N_\alpha(i)}\left(1-n_{\ell\to i}^{[\alpha]}\right)\right)\\ &\times\left\{1-\prod_{\alpha'\neq\alpha}\left[1-s_{i\alpha'}+s_{i\alpha'}\prod_{\ell\in N_{\alpha'}(i)}\left(1-n_{\ell\to i}^{[\alpha']}\right)\right]\right\}.(3)\end{aligned}$$

The average number S of replica nodes belonging to the RMCGC is computed as

$$S=\frac{1}{MN}\sum_{\alpha=1}^{M}\sum_{i=1}^{N}\sigma_{i\alpha}.\qquad(4)$$

The system of Eqs. (1), (2), (3), and (4) represents a complete mathematical framework to estimate the average size of the RMCGC for a given network and a given initial configuration of damage. The solution can be obtained by first iterating Eqs. (1) and (2) to obtain the values of the messages $n_{i\to j}^{[\alpha]}$. Those values are then plugged into Eqs. (3) to compute the values of the variables $s_{i\alpha}$, and finally these variables are used into Eq. (4) to estimate the average size of the RMCGC. We stress that, being valid for a given network and for a given configuration of damage, the values of the variables $n_{i\to j}^{[\alpha]}$ and $s_{i\alpha}$ are either 0 or 1. The variables $v_{i\to j}$ can assume instead integer values in the range $[0,M]$. The mathematical framework works properly also in presence of edge overlap among layers. This is an important feature that can change dramatically change the robustness properties of multilayer networks [19, 27–30].

IV. MULTILAYER NETWORKS WITHOUT LINK OVERLAP

A. General results

1. *Simplification of the message-passing equations on a single realization of the initial damage*

In absence of link overlap, a given pair of nodes i and j may be linked exclusively along a single layer α. Nontrivial messages potentially different from zero will therefore exist only on a specific layer for every pair of connected nodes i and j. It can be easily seen that the message-passing Eqs. (1) and (2) reduce to

$$\begin{aligned}n_{i\to j}^{[\alpha]} &= s_{i\alpha}s_{j\alpha}a_{ij}^{[\alpha]}\left[1-\prod_{\ell\in N_\alpha(i)\setminus j}\left(1-n_{\ell\to i}^{[\alpha]}\right)\right]\\ &\times\left\{1-\prod_{\alpha'\neq\alpha}\left[1-s_{i\alpha'}+s_{i\alpha'}\prod_{\ell\in N_{\alpha'}(i)}\left(1-n_{\ell\to i}^{[\alpha']}\right)\right]\right\}.(5)\end{aligned}$$

We further notice that in this situation the result of the message-passing algorithm does not change if we consider messages that depend exclusively on the state $s_{i\alpha}$ of the node i that sends the message. Even if we drop the factor $s_{j\alpha}$ in Eq. (5), the message will be allowed anyways to propagate further at the next iteration step, if the replica node (j,α) is not initially damaged. Therefore, we can further simplify Eq. (5) and consider

$$\begin{aligned}n_{i\to j}^{[\alpha]} &= s_{i\alpha}a_{ij}^{[\alpha]}\left[1-\prod_{\ell\in N_\alpha(i)\setminus j}\left(1-n_{\ell\to i}^{[\alpha]}\right)\right]\\ &\times\left\{1-\prod_{\alpha'\neq\alpha}\left[1-s_{i\alpha'}+s_{i\alpha'}\prod_{\ell\in N_{\alpha'}(i)}\left(1-n_{\ell\to i}^{[\alpha']}\right)\right]\right\}.(6)\end{aligned}$$

Eqs. (6) replace Eqs. (1) and (2) in the case of a multilayer network without link overlap. The rest of the framework is identical, so that Eqs. (3) and (4) remain unchanged.

2. Message-passing equations for random realizations of the initial damage

Eqs. (6), (3), and (4) determine the average size of the RMCGC in a multilayer network without link overlap for a given realization of the initial damage $\{s_{i\alpha}\}$. These equations can be, however, extended to make predictions in the case of a random realization of the initial damage when the replica nodes are damaged independently with probability $1-p$, i.e,. such that the the initial damage $\{s_{i\alpha}\}$ is a random configuration obeying the probability distribution

$$\hat{\mathcal{P}}(\{s_{i\alpha}\}) = \prod_{i=1}^{N}\prod_{\alpha=1}^{M} p^{s_{i\alpha}}(1-p)^{1-s_{i\alpha}}. \tag{7}$$

To this end, we denote the probability that node i sends a positive message to node j in layer α by $\hat{n}_{i\to j}^{[\alpha]} = \left\langle n_{i\to j}^{[\alpha]}\right\rangle$, and the probability that the replica node (i,α) belongs to the RMCGC by $\hat{\sigma}_{i\alpha} = \langle \sigma_{i\alpha}\rangle$. The message-passing algorithm determining the values of $\hat{n}_{i\to j}^{[\alpha]}$ and $\hat{\sigma}_{i\alpha}$ is given by

$$\begin{aligned}
\hat{n}_{i\to j}^{[\alpha]} &= a_{ij}^{[\alpha]} p \left[1 - \prod_{\ell\in N_\alpha(i)\setminus j}\left(1-\hat{n}_{\ell\to i}^{[\alpha]}\right)\right] \\
&\times \left\{1-\prod_{\alpha'\neq\alpha}\left[1-p+p\prod_{\ell\in N_{\alpha'}(i)}\left(1-\hat{n}_{\ell\to i}^{[\alpha']}\right)\right]\right\}, \\
\hat{\sigma}_{i\alpha} &= p\left(1-\prod_{\ell\in N_\alpha(i)}\left(1-\hat{n}_{\ell\to i}^{[\alpha]}\right)\right) \\
&\times \left\{1-\prod_{\alpha'\neq\alpha}\left[1-p+p\prod_{\ell\in N_{\alpha'}(i)}\left(1-\hat{n}_{\ell\to i}^{[\alpha']}\right)\right]\right\}.
\end{aligned} \tag{8}$$

This algorithm can be applied to a given network, and provides the average number of replica nodes S belonging to the RMCGC for a random realization of the initial damage obeying Eq. (7). Specifically the value of $\hat{S}$ is related to $\hat{\sigma}_{i\alpha}$ by

$$\hat{S} = \frac{1}{MN}\sum_{i=1}^{N}\sum_{\alpha=1}^{M}\hat{\sigma}_{i\alpha}. \tag{9}$$

3. Message-passing equations for random multilayer networks

A multilayer network where every layer is a sparse network generated according to the configuration model is a major example of a multilayer network without link overlap in the limit of large network sizes. It is therefore natural and important to characterize the RMCGC in this case. We assume that every network layer G_α is a random graph taken from the probability distribution

$$\mathcal{P}^{[\alpha]}(G_\alpha) = \frac{1}{Z}\prod_{i=1}^{N}\delta\left(k_i^{[\alpha]}, \sum_{j=1}^{N} a_{ij}^{[\alpha]}\right), \tag{10}$$

where $k_i^{[\alpha]}$ indicates the preimposed degree of node i in layer α, $\delta(x,y)=1$ if $x=y$ and $\delta(x,y)=0$, otherwise, and Z is the normalization factor indicating the total number of networks in the ensemble. Averaging over the network ensemble allows us to translate the message-passing equations into simpler expressions for the characterization of the percolation transition.

Let us consider a random multilayer network obeying the probability of Eq. (10), and a random realization of the initial damage described by the probability of Eq. (7). The average message in layer α, namely $S'_\alpha = \left\langle \hat{n}_{i\to j}^{\alpha} | a_{ij}^{[\alpha]}=1\right\rangle$, and the average number of replica nodes of layer α that are in the RMCGC, denoted by $S_\alpha = \langle \hat{\sigma}_{i,\alpha}\rangle$, obey the equations

$$\begin{aligned}
S_\alpha &= p\sum_{\mathbf{k}} P(\mathbf{k})\left[1-(1-S'_\alpha)^{k^{[\alpha]}}\right] \\
&\quad \left\{1-\prod_{\alpha'\neq\alpha}\left[1-p+p(1-S'_{\alpha'})^{k^{[\alpha']}}\right]\right\}, \\
S'_\alpha &= p\sum_{\mathbf{k}} \frac{k^\alpha}{\langle k^\alpha\rangle} P(\mathbf{k})\left[1-(1-S'_\alpha)^{k^{[\alpha]}-1}\right], \\
&\quad \times\left\{1-\prod_{\alpha'\neq\alpha}\left[1-p+p(1-S'_{\alpha'})^{k^{[\alpha']}}\right]\right\},
\end{aligned} \tag{11}$$

where $P(\mathbf{k})$ indicates the probability that a generic node i has degrees $\mathbf{k}_i = \mathbf{k}$, i.e. $(k_i^{[1]}, k_i^{[2]}, \ldots, k_i^{[M]}) = (k^{[1]}, k^{[2]}, \ldots, k^{[M]})$.

If there are no correlations between the degrees of a node in different layers, the degree distribution $P(\mathbf{k})$ can be factorized as

$$P(\mathbf{k}) = \prod_{\alpha} P^{[\alpha]}(k^{[\alpha]}), \tag{12}$$

where $P^{[\alpha]}(k)$ is the degree distribution in layer α. In this case, Eqs. (11) can be expressed in terms of the generating function of the degree distribution in each layer.

Specifically, we have

$$
\begin{aligned}
S_\alpha &= p\left[1-H_0^{[\alpha]}(1-S'_\alpha)\right] \\
&\quad \left\{1-\prod_{\alpha'\neq\alpha}\left[1-p+p\,H_0(1-S'_{\alpha'})\right]\right\}, \\
S'_\alpha &= p\left[1-H_1^{[\alpha]}(1-S'_\alpha)\right] \\
&\quad \left\{1-\prod_{\alpha'\neq\alpha}\left[1-p+p\,H_0^{[\alpha']}(1-S'_{\alpha'})\right]\right\}, \quad (13)
\end{aligned}
$$

where the generating functions $H_0^{[\alpha]}(z)$ and $H_1^{[\alpha]}(z)$ of the degree distribution $P^{[\alpha]}(k)$ of layer α are given by

$$
\begin{aligned}
H_0^{[\alpha]}(x) &= \sum_k P^{[\alpha]}(k)\,x^k, \\
H_1^{[\alpha]}(x) &= \sum_k \frac{k}{\langle k^{[\alpha]}\rangle} P^{[\alpha]}(k)\,x^{k-1}. \quad (14)
\end{aligned}
$$

Finally the average number S of replica nodes in the RMCGC is given by

$$
S = \frac{1}{M}\sum_\alpha S_\alpha. \quad (15)
$$

If we consider the case of equally distributed Poisson layers with average degree z, we have that Eq. (12) is

$$
P^{[\alpha]}(k) = \frac{1}{k!}z^k e^{-z} \quad (16)
$$

for every layer $\alpha = 1, 2, \ldots, M$. Then, using Eqs. (13), one can show that $S'_\alpha = S_\alpha = S$, $\forall\alpha$, and S is determined by the equation

$$
S = p\left(1-e^{-zS}\right)\left\{1-\left[1-p+pe^{-zS}\right]^{M-1}\right\}. \quad (17)
$$

This equation has always the trivial solution $S = 0$. In addition, a nontrivial solution $S > 0$ indicating the presence of the RMCGC, emerges at a hybrid discontinuous transition characterized by a square root singularity, on a line of points $p = p_c(z)$, determined by the equations

$$
\begin{aligned}
h_{z,p}(S_c) &= 0, \\
\left.\frac{dh_{z,p}(S)}{dS}\right|_{S=S_c} &= 0, \quad (18)
\end{aligned}
$$

where

$$
\begin{aligned}
h_{z,p}(S) &= S - p(1-e^{-zS}) \\
&\quad \times\left\{1-\left[1-p+pe^{-zS}\right]^{M-1}\right\} = 0. \quad (19)
\end{aligned}
$$

For $p > p_c$ there is a RMCGC, for $p \leq p_c$ there is no RMCGC. The entity of the discontinuous jump at $p = p_c$ in the fraction S of replica nodes in the RMCGC is given by $S = S_c$. The percolation threshold p_c as a function of the average degree z of the network is plotted in Figure 2 for $M = 2, 3, 4, 5$. It is shown that as the number of layers M increases the percolation threshold decreases for every value of the average degree z. Additionally also the discontinuous jump S_c decreases as the number M of layer increases for very given average degree z (see Figure 3). Therefore as the number of layers increases the multilayer networks becomes more robust.

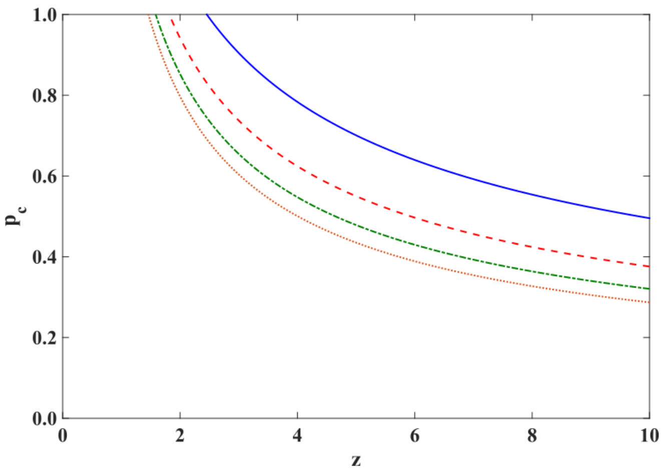

FIG. 2: The percolation threshold p_c is plotted versus the average degree z of each layer for Poisson multilayer networks with $M = 2, 3, 4, 5$ layers indicated respectively with with blue solid, red dashed, green dot-dashed and orange dotted lines.

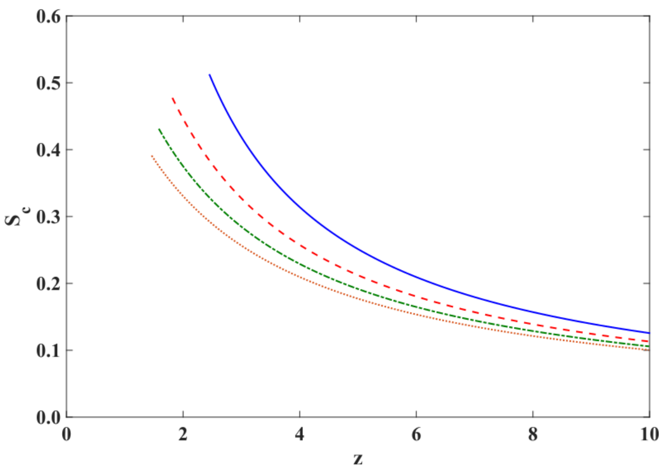

FIG. 3: The discontinuous jump $S_c = S(p_c)$ of the RMCGC at the percolation threshold $p = p_c$, is plotted versus the average degree z of each layer for Poisson multilayer networks with $M = 2, 3, 4, 5$ layers indicated respectively with blue solid, red dashed, green dot-dashed and orange dotted lines.

B. Comparison between the RMCGC and the MCGC

In this section, we compare the robustness of multilayer networks in presence of ordinary interdependencies and in presence of redundant interdependencies. To take a concrete example, we consider the case of a multilayer network with M Poisson layers, each layer having the same average degree z. In this case the fraction S of replica nodes in the RMCGC is given by the solution of Eqs. (17) while the fraction of replica nodes in the MCGC is given by

$$S = \tilde{p}\left(1 - e^{-zS}\right)^{M}. \tag{20}$$

In Eq. (20), it is assumed that every replica node (i, α) of a given node i is damaged simultaneously (with probability $\tilde{f} = 1 - \tilde{p}$). On the contrary, in presence of redundant interdependencies it is natural to assume that the initial damage is inflicted to each replica node independently (with probability $f = 1 - p$). Therefore, in order to compare the robustness of the multilayer networks in presence and in absence of redundant interdependencies, we set $p = \tilde{p} = 1$, i.e., replica nodes are not initially damaged, and compare the critical value of the average degree $z = z^\star$ at which the percolation transition occurs respectively for the RMCGC and for the MCGC. Additionally we will characterize also the size $S = S^\star$ of the jump in the size of the RMCGC and the MCGC at the percolation transition. In Fig. 4, we display the values of $z^\star$ and $S^\star$ as a function of the number of layers M for the RMCGC and the MCGC. For $M = 2$, the two models give the same results as they are identical. For $M > 2$, differences arise. In presence of redundant interdependencies, multilayer networks become increasingly more robust as the number M of layers increases. This phenomenon is apparent from the fact that the RMCGC emerges for multilayer networks with an average degree of their layers $z^\star$ which decreases as the number of layers M increases. On the contrary, in ordinary percolation the value of $z^\star$ for the emergence of the MCGC is an increasing function of M. Additionally, the size of the discontinuous jumps $S^\star$ at the transition point decreases with M for the RMCGC, while increases with M for the MCGC showing that the avalanches of failures have a reduced size for the RMCGC.

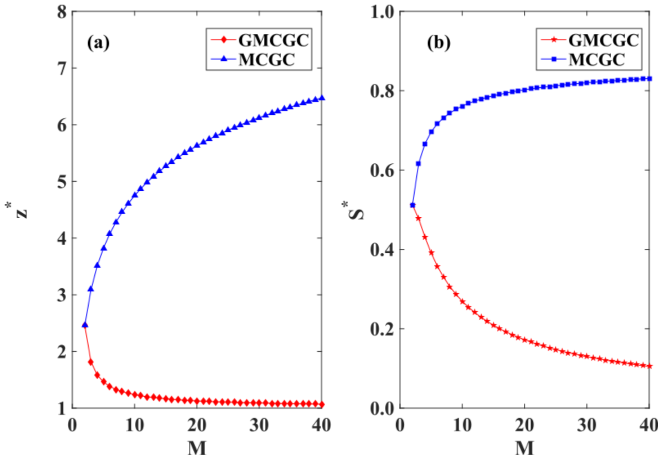

FIG. 4: Comparison between the MCGC and the RMCGC models in Poisson multilayer networks. (a) Critical value $z^\star$ of the average degree as a function of the number of network layers M. Results for the RMCGC model are displayed as red diamonds. Results for the MCGC model are denoted by blue triangles. (b) Height of the jump $S^\star$ at the transition point as a function of the number of network layers.

C. Comparison with numerical simulations

In this section, we compare the results obtained with Eqs. (1), (2), (3), and (4) on a single instance of damage with the predictions the message-passing algorithm described in Eq. (13) characterizing the size S of the RMCGC in an ensemble of networks. Specifically, we consider the case of a multilayer network with $M = 3$ Poisson layers with the same average degree z. In order to draw the percolation diagram for single instances of initial damage as a function of the probability of damage $1 - p$, we associate each replica node (i, α) with a random variable $r_{i\alpha}$ drawn from a uniform distribution and we set

$$s_{i\alpha} = \begin{cases} 1 & \text{if } r_{i\alpha} \leq p \\ 0 & \text{if } r_{i\alpha} > p \end{cases} \tag{21}$$

Fig. 5 displays the comparison between the two approaches, showing an almost perfect agreement between them. Additionally in Fig. 6, we compare simulation results averaged over several realizations of the initial damage and several instances of the multilayer network model with the theoretical predictions given by the numerical solution of Eqs. (11)-(17), obtaining a very good agreement.

V. MULTILAYER NETWORKS WITH LINK OVERLAP

A. Link overlap, multilinks and multidegree

In isolated networks, two nodes can be either connected or not connected. In multilayer networks instead, the complexity of the structure greatly increases as the ways in which a generic pair of nodes can be connected is given by 2^M possibilities. A very convenient way of accounting for all the possibilities with a compact notation is to use the notion of multilink among pairs of nodes [5, 31]. Multilinks $\vec{m} = \left(m^{[1]}, m^{[2]}, \ldots, m^{[M]}\right)$ with $m^{[\alpha]} = 0, 1$, describe any of the possible patterns of connections between pairs of nodes in a multilayer network with M layers. Specifically, $m^{[\alpha]} = 1$ indicates that a

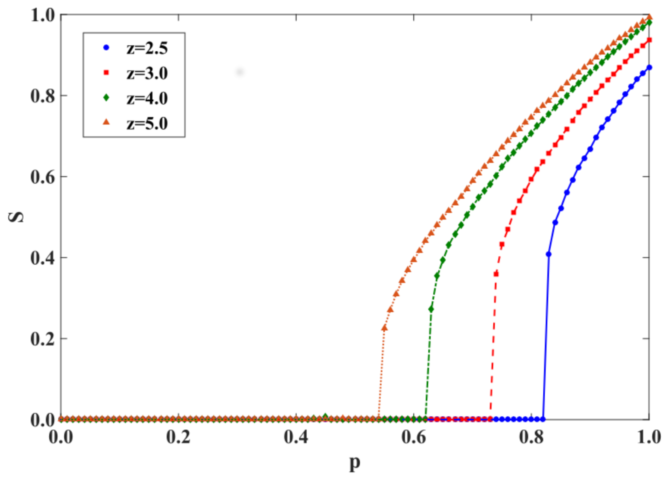

FIG. 5: Comparison between simulation results of the RMCGC for a multilayer network with $M = 3$ Poisson layers Poisson with average degree z and no link overlap, and the message-passing results over single network realization and given configuration damage. We consider different values of the average degree $z = 2.5, 3.0, 4.0, 5.0$. Points indicate results of numerical simulations: blue circles ($z = 2.5$), red squares ($z = 3.0$), green diamonds ($z = 4.0$), and orange triangles ($z = 5.0$). Message-passing predictions are denoted by lines with the same color scheme used for numerical simulations. Simulations results are performed on a single instance of a multilayer network with $N = 10^4$ nodes.

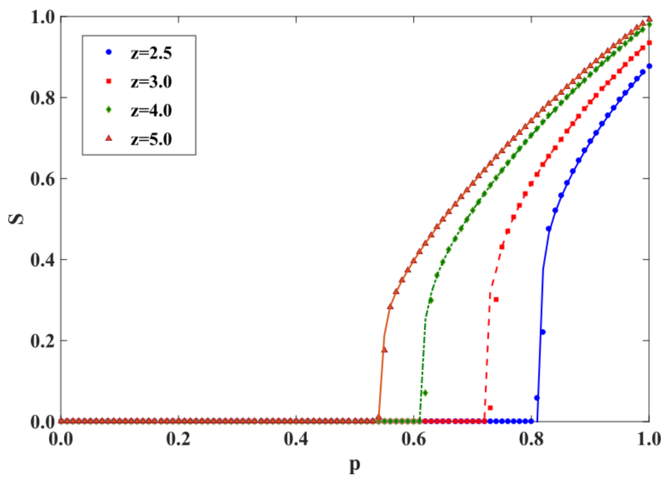

FIG. 6: Same as in Fig. 5, but for averages over 20 instances of the multilayer network model and configurations of random initial damage.

connection exists in layer α, whereas $m^{[\alpha]} = 0$ indicates that the connection in layer α does not exists. In particular, we can say that, in a multilayer network with M layers, two nodes i and j are connected by the multilink

$$\vec{m}_{ij} = (a_{ij}^{[1]}, a_{ij}^{[2]}, \ldots, a_{ij}^{[M]}). \tag{22}$$

In order to distinguish the case in which two nodes are not connected in any layer with the case in which in at least one layer the nodes are connected, we distinguish between the trivial multilink $\vec{m} = \vec{0}$ and the nontrivial multilinks $\vec{m} \neq \vec{0}$. The trivial multilink $\vec{m} = \vec{0}$ indicates the absence of any sort of link between the two nodes.

Using the concept of multilinks, one can define multiadjacency matrices $\mathbf{A}^{\vec{m}}$ whose element $A_{ij}^{\vec{m}}$ indicates whether ($A_{ij}^{\vec{m}} = 1$) or not ($A_{ij}^{\vec{m}} = 0$) a node i is connected to node j by a multilink $\vec{m}$. The matrix elements $A_{ij}^{\vec{m}}$ of the multiadjacency matrix $\mathbf{A}^{\vec{m}}$ are given by

$$A_{ij}^{\vec{m}} = \prod_{\alpha=1}^{M} \delta\left(m^{[\alpha]}, a_{ij}^{[\alpha]}\right). \tag{23}$$

Using multiadjacency matrices, it is straightforward to define multidegrees [5, 31]. The multidegree of node i indicated as $k_i^{\vec{m}}$ is the sum of rows (or columns) of the multiadjacency matrix $\mathbf{A}^{\vec{m}}$, i.e.,

$$k_i^{\vec{m}} = \sum_j A_{ij}^{\vec{m}}, \tag{24}$$

and indicates how many multilinks $\vec{m}$ are incident to node i.

Using a multidegree sequence $\{k_i^{\vec{m}}\}$, it is possible to build multilayer network ensembles that generalize the configuration model. This way, overlap of links is fully preserved by the randomization of the multilayer network. These ensembles are specified by the probability $\tilde{\mathcal{P}}(\vec{G})$ attributed to every multilayer network $\vec{G}$ of the ensembles, where $\tilde{\mathcal{P}}(\vec{G})$ is given by

$$\tilde{\mathcal{P}}(\vec{G}) = \frac{1}{\tilde{Z}} \prod_{i=1}^{N} \prod_{\vec{m} \neq \vec{0}} \delta\left(k_i^{\vec{m}}, \sum_{j=1}^{N} A_{ij}^{\vec{m}}\right), \tag{25}$$

with $\tilde{Z}$ normalization constant equal to the number of multilayer networks with given multidegree sequence.

B. General discussion of the message passing equations for the RMCGC

Our goal here is to generalize the message-passing algorithm already given by Eqs. (1), (2), (3), and (4) for a generic single instance of a multilayer network and single realization of initial damage to the cases of (i) random multilayer networks with given multidegree sequence and/or (ii) random realizations of the initial damage. The extensions for both cases has been already considered for the case of multilayer networks without link overlap. In presence of link overlap, however, the approach is much more cumbersome. For two nodes i and j in fact, the messages $n_{i\to j}^{[\alpha]}$ given by Eq. (1) and sent from node i to node j over the different layers $\alpha = 1, 2 \ldots M$ are correlated because they all depend on the value of the variable $v_{i\to j}$ given by Eq. (2). Such correlations require particular care when averaging the messages to treat the percolation transition for random initial damages. Similar technical challenges are also present in the treatment of the standard MCGC model where interdependencies are not redundant [19, 27]. In presence of redundant interdependencies there is an additional precaution that needs to be taken. In fact, the messages $n_{i\to j}^{[\alpha]}$ are explicitly dependent on the state of all replicas (j,α') of node j. This state is indicated by the variables $\vec{s}_j=(s_{j1},s_{j2},\ldots,s_{j\alpha'},\ldots s_{jM})$ where $s_{j\alpha'}$ specifies whether the replica node (j,α') is initially damaged or not. As a consequence of this property, when averaging over random realizations of initial damage, message-passing equations are written in terms of the messages $\hat{\sigma}_{i\to j}^{\vec{m}_{ij},\vec{n}}(\vec{s}_j)$ explicitly accounting for the probability that node i is sending to node j the set of messages $\vec{n}=(n_{i\to j}^{[1]},n_{i\to j}^{[2]}\ldots n_{i\to j}^{[\alpha]},\ldots n_{i\to j}^{[M]})$, given that node j is in state $\vec{s}_j$ and node i and node j are connected by a multilink $\vec{m}=\vec{m}_{ij}$. We have derived these equations for a general multilayer network with M layers. However, the message-passing algorithm has a very long expression. To make the paper more readable, we decided to place the exact treatment of the general case in the SM, and consider here only the special case of ensembles of random multilayer networks with overlap. For these ensembles in fact, the message-passing equations are written in terms of average messages sent between nodes with given multilinks $\vec{m}$, i.e., $S^{\vec{m},\vec{n}}(\vec{s}_j)=\left\langle\hat{\sigma}_{i\to j}^{\vec{m}_{ij},\vec{n}}(\vec{s}_j)|\vec{m}_{ij}=\vec{m}\right\rangle$, and the equations greatly simplify. Two specific cases of multilayer network ensembles are discussed below, for the cases of $M=2$ and $M=3$ layers.

C. Ensembles of multilayer networks link overlap and $M=2$ layers

In this case, every replica node is in the RMCGC if and only if also its interdependent node in the other layer is in the RMCGC. Therefore, the only messages that are different from zero are the messages $S^{\vec{m},\vec{n}}(\vec{s}_j=(1,1))$ sent to nodes j in state $\vec{s}_j=(1,1)$. Specifically, we consider the case of a random multilayer network with Poisson multidegree distributions characterized by the averages

$$\begin{aligned}\langle k^{(1,1)}\rangle &= z_2,\\ \langle k^{(0,1)}\rangle &= \langle k^{(1,0)}\rangle = z_1.\end{aligned}\tag{26}$$

The messages $S^{\vec{m},\vec{n}}(\vec{s}_j=(1,1))$ only depend on the multiplicity of overlap of the multilinks $\vec{m}$ and $\vec{n}$ given respectively by

$$\begin{aligned}\mu &= \sum_{\alpha=1}^{M} m^{[\alpha]},\\ \nu &= \sum_{\alpha=1}^{M} n^{[\alpha]}.\end{aligned}\tag{27}$$

The fraction S of replica nodes in the RMCGC is determined by the variables

$$x_{\mu,\nu}=S^{\vec{m},\vec{n}}(\vec{s}_j=(1,1)).\tag{28}$$

The value of $x_{2,2}$ indicates the probability that node i to sends a message $\vec{n}=(1,1)$ to its neighbor j with $\vec{s}_j=(1,1)$ connected by a multilink $\vec{m}=(1,1)$. This fact occurs if and only if node i has both replica nodes that are not initially damaged (which occurs with probability p^2) and if at least one positive message in each layer α is reaching node i from neighbors different from j. The value of $x_{1,1}$ indicates the probability that node i sends a message $\vec{n}=(1,0)$ to its neighbor j with $\vec{s}_j=(1,1)$ connected by a multilink $\vec{m}=(1,0)$ or equivalently sends a message $\vec{n}=(0,1)$ to its neighbor j with $\vec{s}_j=(1,1)$ connected by a multilink $\vec{m}=(0,1)$. This fact occurs if and only if node i has both replica nodes that are not initially damaged (which occurs with probability p^2) and if at least one positive message in each layer α is reaching node i from neighboring nodes different from j. The latter is a necessary condition to have $v_{i\to j}=2$. The value $x_{2,1}$ indicates the probability that node i is sending a message $\vec{n}=(1,0)$ to its neighbor j in state $\vec{s}_j=(1,1)$ and connected by a multilink $\vec{m}=(1,1)$ or equivalently sends a message $\vec{n}=(0,1)$ to its neighbor j in state $\vec{s}_j=(1,1)$ and connected by a multilink $\vec{m}=(1,1)$. This fact occurs if only if node i has both replica nodes that are not initially damaged (which occurs with probability p^2) and if at least one positive message is reaching node i in the layer for which $n^{[\alpha]}=1$ from neighbors different from j and no positive message is reaching node i in the layer where $n^{[\alpha]}=1$ from neighboring nodes different from node j. Finally, S is the probability that a replica node (i,α) is in the RMCGC which implies that (i) it is not initially damaged, (ii) its replica node in the other layer is not initially damaged, and (iii) at least one positive message reaches node i in both layers.

The values of the variables $x_{\mu,\nu}$ and S are therefore determined by the following set of equations

$$\begin{aligned}x_{2,2} =&\ p^2\left[1-2e^{-z_1x_{1,1}-z_2(x_{2,2}+x_{2,1})}\right.\\ &\left.+e^{-2z_1x_{1,1}-z_2(x_{2,2}+2x_{2,1})}\right]\\ x_{2,1} =&\ p^2\left[e^{-z_1x_{1,1}-z_2(x_{2,2}+x_{2,1})}\right.\\ &\left.-e^{-2z_1x_{1,1}-z_2(x_{2,2}+2x_{2,1})}\right]\\ S =&\ x_{1,1}=x_{2,2}\end{aligned}\quad.\tag{29}$$

These equations are the same equations as those that determine the value of the MCGC as long as the fact that the damage in each replica node is independent is taken into account, which can be done by making the substitution $p^2\to p$ [19, 27].

D. Ensembles of multilayer networks link overlap and $M=3$ layers

We consider now the case of a random multilayer network with $M=3$ layers. The network has Poisson multidegree distributions and averages given by

$$\begin{aligned}\langle k^{(1,1,1)}\rangle &= z_3,\\ \langle k^{(1,1,0)}\rangle &= \langle k^{(1,0,1)}\rangle = \langle k^{(0,1,1)}\rangle = z_2,\\ \langle k^{(1,0,0)}\rangle &= \langle k^{(0,1,0)}\rangle = \langle k^{(0,0,1)}\rangle = z_1. \end{aligned} \tag{30}$$

In this case, the messages $S^{\vec{m},\vec{n}}(\vec{s}_j)$ only depend on the multiplicity of overlap of the multilinks $\vec{m}$ and $\vec{n}$ and the number of layers where $s_{j,\alpha}=1$ and $m^{[\alpha]}=1$. Therefore, messages depend only on

$$\begin{aligned}\mu &= \sum_{\alpha=1}^{M} m^{[\alpha]},\\ \nu &= \sum_{\alpha=1}^{M} n^{[\alpha]},\\ \xi &= \sum_{\alpha=1}^{M} s_{j\alpha} m^{[\alpha]}. \end{aligned} \tag{31}$$

The fraction of replica nodes in the RMCGC S is deter-

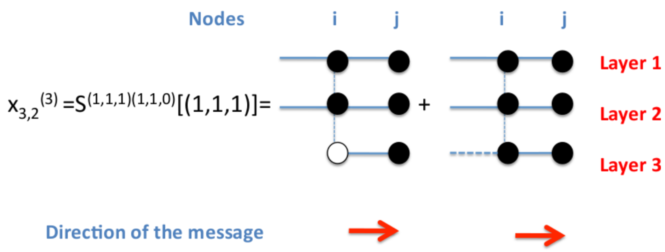

FIG. 7: Example of a diagrammatic representation of the equations determining $x^{(3)}_{3,2} = S^{(1,1,1),(1,1,0)}[(1,1,1)]$ in a multilayer network with $M=3$ layers and $\langle k^{(1,1,1)}\rangle = z_3$, $\langle k^{(1,1,0)}\rangle = \langle k^{(1,0,1)}\rangle = \langle k^{(0,1,1)}\rangle = z_2$ and $\langle k^{(1,0,0)}\rangle = \langle k^{(0,1,0)}\rangle = \langle k^{(0,0,1)}\rangle = z_1$. Filled circles indicate initially undamaged replica nodes $s_{i\alpha}=1$, whereas empty circles indicate initially damaged replica replica nodes $s_{i\alpha}=0$. The message are sent along the direction indicated by the arrows. A solid line reaching node i in layer α indicates that at least one positive message is reaching node i from nodes different from node j in layer α. Dotted lines joining node i in layer α indicate that no positive message reaches node i from nodes different from node j in layer α. A solid (dotted) line between node i and node j in layer α indicates $m^{[\alpha]}=1$ ($m^{[\alpha]}=0$).

mined by the variables

$$x^{(\xi)}_{\mu,\nu} = S^{\vec{m},\vec{n}}(\vec{s}_j). \tag{32}$$

Let us explicitly describe the equations that one of these variables needs to satisfy, and introduce a symbolic way to describe the equations. Specifically, we consider $x^{(3)}_{3,2}$ as the probability $S^{(1,1,1),(1,1,0)}((1,1,1))$ that a node i, connected to a node j by a multilink $\vec{m}=(1,1,1)$, sends to node j a message $\vec{n}=(1,1,0)$ provided that node j is in the state $\vec{s}_j=(1,1,1)$ (see Fig. 7). This probability is equal to the sum of (i) the probability that node i is in the state $\vec{s}_i=(1,1,0)$ [which occurs with probability $(1-p)p^2$] and it sends the message $\vec{n}=(1,1,0)$ to node j and (ii) the probability that node i is in the state $\vec{s}_i=(1,1,1)$ (which occur with probability p^3) and sends the same message to node j. Node i sends the message $\vec{n}=(1,1,0)$ only if the following conditions are met:

(i) if node i is in the state $\vec{s}_i=(1,1,0)$, node i must receive at least one positive message from nodes different from node j in layers $\alpha=1$ and $\alpha=2$.

(ii) if node i is in the state $\vec{s}_i=(1,1,1)$, node i must receive at least one positive message from nodes different from node j in layers $\alpha=1$ and $\alpha=2$ and must not receive any positive message from nodes different from node j in layer $\alpha=3$.

These requirements are summarized by the diagram of Fig. 7. Diagrams that describe the equations to determine the value of all the other variables $x^{(\xi)}_{\mu,\nu}$ are presented in Fig. 8. These equations read as

$$\begin{aligned}
x^{(3)}_{3,3} &= p^3\left[1-3h_{1,3}+3h_{2,3}-h_{3,3}\right]\\
x^{(3)}_{3,2} &= p^2(1-p)\left[1-2h_{1,2}+h_{2,2}\right]+p^3\left[h_{1,3}-2h_{2,3}+h_{3,3}\right]\\
x^{(2)}_{2,2} &= p^2(1-p)\left[1-2h_{1,2}+h_{2,2}\right]+p^3\left[1-2h_{1,3}+h_{2,3}\right]\\
x^{(2)}_{3,2} &= x^{(2)}_{2,2}\\
x^{(1)}_{1,1} &= 2p^2(1-p)\left[1-2h_{1,2}+h_{2,2}\right]\\
&\quad +p^3\left[1-h_{1,3}-h_{2,3}+h_{3,3}\right]\\
x^{(2)}_{2,1} &= p^2(1-p)\left[h_{1,2}-h_{2,2}\right]+p^2(1-p)\left[1-2h_{1,2}+h_{2,2}\right]\\
&\quad +p^3\left[h_{1,3}-h_{2,3}\right]\\
x^{(1)}_{2,1} &= x^{(1)}_{11}\\
x^{(2)}_{3,1} &= x^{(2)}_{2,1}\\
x^{(3)}_{3,1} &= 2p^2(1-p)\left[h_{1,2}-h_{2,2}\right]+p^3\left[h_{2,3}-h_{3,3}\right]\\
S &= x^{(1)}_{1,1}
\end{aligned} \tag{33}$$

where

$$\begin{aligned}
h_{1,3} &= e^{-z_1 x^{(1)}_{1,1}-z_2(2x^{(2)}_{2,2}+2x^{(2)}_{2,1})-z_3(x^{(3)}_{3,3}+2x^{(3)}_{3,2}+x^{(3)}_{3,1})}\\
h_{2,3} &= e^{-2z_1 x^{(1)}_{1,1}-z_2(3x^{(2)}_{2,2}+4x^{(2)}_{2,1})-z_3(x^{(3)}_{3,3}+3x^{(3)}_{3,2}+2x^{(3)}_{3,1})}\\
h_{3,3} &= e^{-3z_1 x^{(1)}_{1,1}-z_2(3x^{(2)}_{2,2}+6x^{(2)}_{2,1})-z_3(x^{(3)}_{3,3}+3x^{(3)}_{3,2}+3x^{(3)}_{3,1})}\\
h_{1,2} &= e^{-z_1 x^{(1)}_{1,1}-z_2(x^{(2)}_{2,2}+x^{(2)}_{2,1}+x^{(1)}_{2,1})-z_3(x^{(2)}_{3,2}+x^{(2)}_{3,1})}\\
h_{2,2} &= e^{-2z_1 x^{(1)}_{1,1}-z_2(x^{(2)}_{2,2}+2x^{(2)}_{2,1}+2x^{(1)}_{2,1})-z_3(x^{(2)}_{3,2}+2x^{(2)}_{3,1})}
\end{aligned} \tag{34}$$

We note that, in absence of overlap, i.e., for $z_1=z$, $z_2=0$ and $z_3=0$, Eqs. (33) reduce Eqs. (17). By defining a suitable order of the variables $x^{(\xi)}_{\mu,\nu}$, it is possible to introduce a vector $\mathbf{x}$ whose elements are the variables $x^{(\xi)}_{\mu,\nu}$, and rewrite the Eqs. (33) in a matrix form as

$$\mathbf{x} = \mathbf{G}(\mathbf{x}). \tag{35}$$

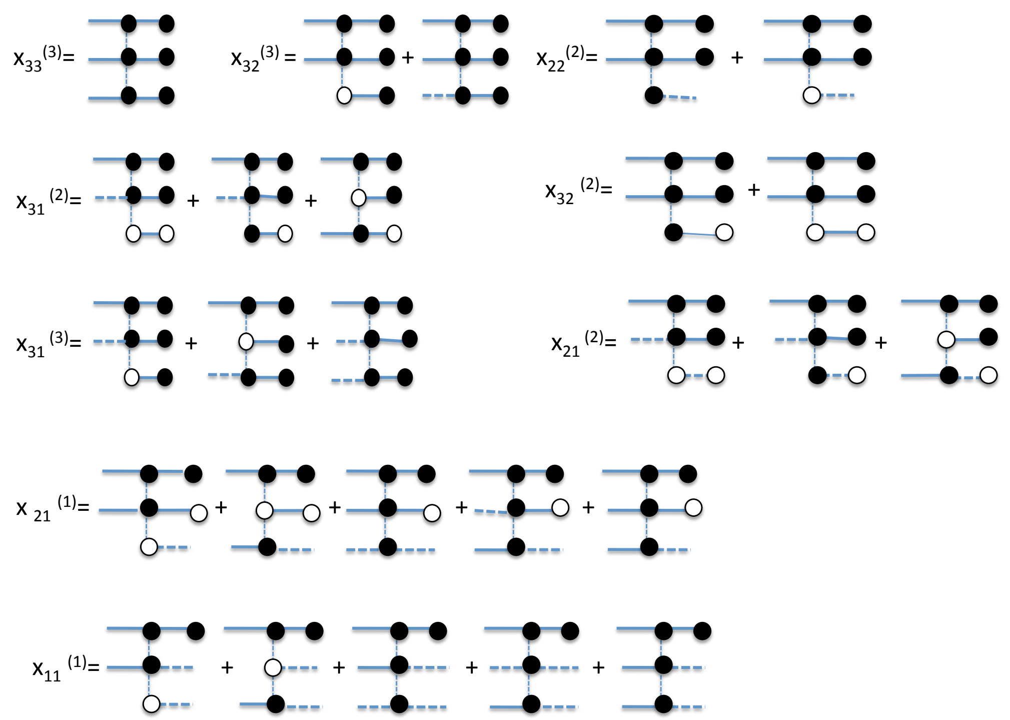

FIG. 8: Diagrams for Eqs. (33) determining $x_{\mu\nu}^{(\xi)}$ in the case of multilayer networks with three layers ($M = 3$) and Poisson multidegree distribution with $\langle k^{(1,1,1)}\rangle = z_3$, $\langle k^{(1,1,0)}\rangle = \langle k^{(1,0,1)}\rangle = \langle k^{(0,1,1)}\rangle = z_2$ and $\langle k^{(1,0,0)}\rangle = \langle k^{(0,1,0)}\rangle = \langle k^{(0,0,1)}\rangle = z_1$.

The hybrid discontinuous phase transition can be found by imposing that the system of Eqs. (35) is satisfied together with the condition that the determinant of the Jacobian $\mathbf{J}$ of $\mathbf{G}(\mathbf{x})$ equals one, that is

$$\mathbf{x} = \mathbf{G}(\mathbf{x}) \quad \text{and} \quad \det \mathbf{J} = 1. \tag{36}$$

E. Simulation results

We perform a comparison between results obtained from the solution of Eqs. (1), (2), (3), and (4), and those obtained from the solution of Eqs. (33) for Poisson multilayer networks composed of $M = 3$ layers and different values of the averages z_1, z_2, and z_3. The percolation transition is studied for single and multiple instances of the multilayer network model as already described in Sec. IV C. Results are presented in Figs. 9 and 10, and they provide clear evidence of a perfect agreement between the two approaches.

VI. CONCLUSIONS

In this paper, we introduced and fully characterized an alternative percolation model for multilayer networks. The model serves to quantify the robustness of networks with redundant interdependencies. According to the model, interdependencies make a system more fragile than it would be by considering each layer independently. This fact is consistent with the original model used to study percolation in multilayer networks [12, 14, 16], and it is apparent from the fact that the transition is abrupt for any number of network layers considered in the interdependent model. On the other hand, redundancy of interdependencies across multiple layers favors system robustness, as the height of the discontinuous jump and the location of the transition point decrease as the number of layers increases. This is a fundamental difference with respect to the model currently adopted to study the robustness of multilayer networks, where instead increasing the number of layers generates more and more fragile networks [16, 19–21]. We believe that having a model where system robustness is augmented by the number of

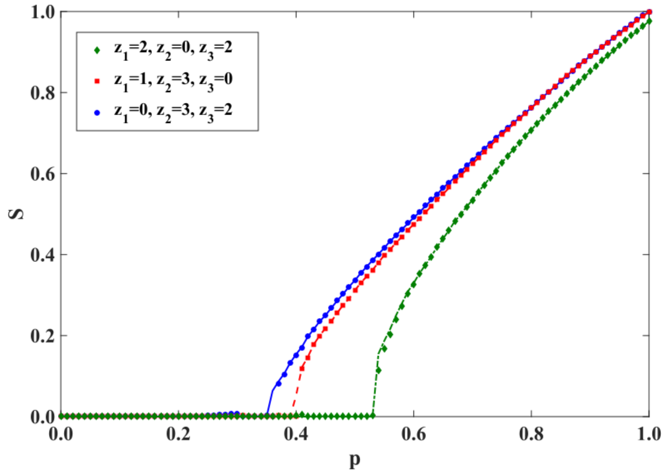

FIG. 9: Comparison between the simulation results and message-passing theory for a multilayer network with $M = 3$ layers and Poisson multidegree distribution with $\langle k^{(1,1,1)} \rangle = z_3$, $\langle k^{(1,1,0)} \rangle = \langle k^{(1,0,1)} \rangle = \langle k^{(0,1,1)} \rangle = z_2$ and $\langle k^{(1,0,0)} \rangle = \langle k^{(0,1,0)} \rangle = \langle k^{(0,0,1)} \rangle = z_1$. We consider here a single network instance and a given configuration of damage. Data are shown for $z_1 = 0, z_2 = 3, z_3 = 2$ (blue), $z_1 = 1, z_2 = 3, z_3 = 0$, (red), and $z_1 = 2, z_2 = 0, z_3 = 2$ (green). Symbols stand for results from numerical simulations, whereas lines represent results for the numerical solution of the message-passing equations. Simulations results are performed on networks with $N = 10^4$ nodes.

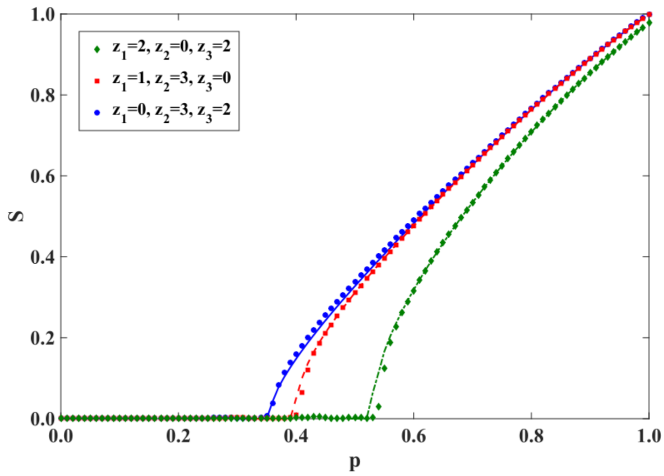

FIG. 10: Same as in Fig. 9, with the difference that results of the simulations stand for averages obtained over 20 instances of the multilayer network models and configurations of the initial damage.

layers is generally more appropriate. Often, new interdependent layers are indeed created to provide backup options. For example, adding a new mode of transportation in a preexisting multimodal transportation system should make the system more reliable against eventual failures. Similarly in a living organism, the development of new types of interactions among constituents should increase the stability of the same organism against possible mutations. In the current setting, the model assumes that the functioning of individual nodes requires that nodes are correctly operating on at least two interdependent layers. The model can be, however, generalized to deal with a variable number of minimal functioning layers to describe more realistic scenarios in specific situations of interest.

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[33] This special setting of a multilayer network is also referred to as multiplex network.

SUPPLEMENTAL MATERIAL

Message passing for given multilayer network and given initial damage

Let us consider a given multilayer network $\vec{G}$ with M layers. Each layer $\alpha = 1, 2, \ldots, M$ of the multilayer network has adjacency matrix $\mathbf{a}^{[\alpha]}$. In this multilayer network, each pair of nodes i and j is connected by a multilink

$$\vec{m}_{ij} = (a_{ij}^{[1]}, a_{ij}^{[2]} \ldots, a_{ij}^{[\alpha]}, \ldots, a_{ij}^{[M]}). \tag{SM1}$$

Any two nodes i and j are connected by a nontrivial multilink is $\vec{m}_{ij} \neq \vec{0}$ implying that at least one link between the two nodes is present across the M layers. We assume that the initial damage configuration is known and that it is given by the set of variables $\{s_{i\alpha}\}$ where $s_{i\alpha}$ indicates if a replica (i,α) is initially damaged ($s_{i\alpha} = 1$) or not ($s_{i\alpha} = 0$). The message passing algorithm given in Sec. III of the main text allows us to determine for any given initial damage configuration, if any replica node (i,α) is in the RMCGC ($\sigma_{i\alpha} = 1$) or not ($\sigma_{i\alpha} = 0$) as long as the multilayer network is locally tree-like. Specifically the variables $\sigma_{i\alpha}$ are determined in terms the set of messages

$$\vec{n}_{i\to j} = (n_{i\to j}^{[1]}, n_{i\to j}^{[2]}, \ldots, n_{i\to j}^{[\alpha]}, \ldots n_{i\to j}^{[M]}) \tag{SM2}$$

going from any node i to any node j joined by a nontrivial multilink $\vec{m}_{ij} \neq \vec{0}$.

The messages $\vec{n}_{i\to j}$ are determined according to the following recursive equation

$$n_{i\to j}^{[\alpha]} = \theta(v_{i\to j}, 2) a_{ij}^{[\alpha]} s_{j\alpha} s_{i\alpha} \left[1 - \prod_{\ell \in N_\alpha(i)\setminus j} \left(1 - n_{\ell\to i}^{[\alpha]}\right)\right], \tag{SM3}$$

where $N_\alpha(i)$ indicates the set of nodes that are neighbor of node i in layer α and where $\theta(x)$ is the step function with values $\theta(v_{i\to j}, 2) = 1$ for $v_{i\to j} \geq 2$ and $\theta(v_{i\to j}, 2) = 0$ for $v_{i\to j} = 0, 1$. Here the variable $v_{i\to j}$ indicates in how many layers node i is connected to the RMCGC assuming that node j also belongs to the RMCGC,

$$v_{i\to j} = \sum_{\alpha=1}^{M} \left\{ s_{i\alpha} \left[1 - \prod_{\ell \in N_\alpha(i)\setminus j} \left(1 - n_{\ell\to i}^{[\alpha]}\right)\right] + s_{i\alpha} s_{j\alpha} a_{ij}^{[\alpha]} \prod_{\ell \in N_\alpha(i)\setminus j} \left(1 - n_{\ell\to i}^{[\alpha]}\right) \right\}. \tag{SM4}$$

Finally the variables $\sigma_{i\alpha}$ are expressed in terms of the messages $\vec{n}_{i\to j}$ and are given by

$$\sigma_{i\alpha} = s_{i\alpha} \left[1 - \prod_{\ell \in N_\alpha(i)} \left(1 - n_{\ell\to i}^{[\alpha]}\right)\right] \left\{1 - \prod_{\alpha' \neq \alpha} \left[1 - s_{i\alpha'} + s_{i\alpha'} \prod_{\ell \in N_{\alpha'}(i)} \left(1 - n_{\ell\to i}^{[\alpha']}\right)\right]\right\}. \tag{SM5}$$

In many situations, however, the initial configuration of the damaged $\{s_{i\alpha}\}$ is not known, and instead it is only known the probability distribution $\hat{\mathcal{P}}(\{s_{i\alpha}\})$ of the initial damage configuration.

In this case, one aims to know the probability $\hat{\sigma}_{i\alpha} = \langle \sigma_{i\alpha} \rangle$ that a replica node (i,α) is in the RMCGC for a random configuration of the initial damage. The value of $\hat{\sigma}_{i\alpha}$, on a locally treelike multilayer network is determined by a distinct message passing algorithm that can be derived from the message passing algorithm valid for single realization of the initial damage, by performing a suitable average of the messages.

Particular care should be taken when one aims to perform this average. In fact $\sigma_{i\alpha}$ depends on all the messages $n_{i\to j}^{[\alpha]}$ sent by node i to node j in all the layers α. These messages are correlated and therefore they cannot be averaged independently.

An alternative formulation of the Eqs. $(SM3)-(SM11)$ provides the necessary framework for deriving in few steps the message passing algorithm to predict $\hat{\sigma}_{i\alpha}$. This alternative formulation is written terms of the variables $\sigma_{i\to j}^{\vec{m},\vec{n}}$ indicating whether ($\sigma_{i\to j}^{\vec{m},\vec{n}} = 1$) or not ($\sigma_{i\to j}^{\vec{m},\vec{n}} = 0$) node i send to node j a message $\vec{n} = \vec{n}_{i\to j}$ given that node j is connected to node i by a multilink $\vec{m} = \vec{m}_{ij}$.

Using Eqs. $(SM3) - (SM4)$ it is easy to see that the value of the variables $\sigma_{i\to j}^{\vec{m},\vec{n}}$ is determined by the following equations:

(a) if $\nu = \sum_{\alpha=1}^{M} n^{[\alpha]} > 1$ and $\vec{m} = \vec{m}_{ij}$,

$$\sigma_{i\to j}^{\vec{m},\vec{n}} = \prod_{\alpha=1}^{M} \left[m^{[\alpha]} s_{j\alpha} s_{i\alpha} - m^{[\alpha]} s_{j\alpha} s_{i\alpha} \prod_{\ell \in N(i)\setminus j} \left(1 - n_{\ell\to i}^{[\alpha]}\right)\right]^{n^{[\alpha]}} \prod_{\alpha=1}^{M} \left[1 - s_{i\alpha} + s_{i\alpha} \prod_{\ell \in N(i)\setminus j} \left(1 - n_{\ell\to i}^{[\alpha]}\right)\right]^{(1-n^{[\alpha]}) m^{[\alpha]} s_{j\alpha}} \tag{SM6}$$

(b) if $\nu=\sum_{\alpha=1}^M n^{[\alpha]}=1$ and $\vec{m}=\vec{m}_{ij}$,

$$\sigma_{i\to j}^{\vec{m},\vec{n}}=\left\{\prod_{\alpha=1}^{M}\left[m^{[\alpha]}s_{j\alpha}s_{i\alpha}-m^{[\alpha]}s_{j\alpha}s_{i\alpha}\prod_{\ell\in N(i)\setminus j}\left(1-n_{\ell\to i}^{[\alpha]}\right)\right]^{n^{[\alpha]}}\right\}\left\{1-\prod_{\alpha'|n^{[\alpha']}=0}\left[1-s_{i\alpha'}+s_{i\alpha'}\prod_{\ell\in N(i)}\left(1-n_{\ell\to i}^{[\alpha']}\right)\right]\right\}, \tag{SM7}$$

(c) if $\nu=\sum_{\alpha=1}^M n^{[\alpha]}=0$ and $\vec{m}=\vec{m}_{ij}$,

$$\sigma_{i\to j}^{\vec{m},\vec{0}}=1-\sum_{\vec{n}\neq\vec{0}}\sigma_{i\to j}^{\vec{m},\vec{n}}, \tag{SM8}$$

where $\vec{n}_{i\to j}$ is determined in terms of the messages $\sigma_{i\to j}^{\vec{m},\vec{n}}$ as

$$\vec{n}_{i\to j}=\mathrm{argmax}_{\vec{n}}\sigma_{i\to j}^{\vec{m}_{ij},\vec{n}}. \tag{SM9}$$

Finally a replica node (i,α) is in the RMCGC ($\sigma_{i\alpha}=1$) or not ($\sigma_{i\alpha}=0$) depending on the messages it receives from its neighbors, i.e.

$$\sigma_{i\alpha}\;=\;s_{i\alpha}\left[1-\prod_{\ell\in N(i)}\left(1-n_{\ell\to i}^{[\alpha]}\right)\right]\left\{1-\prod_{\alpha'\neq\alpha}\left[1-s_{i\alpha'}+s_{i\alpha'}\prod_{\ell\in N(i)}\left(1-n_{\ell\to i}^{[\alpha']}\right)\right]\right\}. \tag{SM10}$$

Message passing algorithm for random damage

By averaging Eqs. $(SM6)-(SM7)-(SM10)$ we can derive the message passing algorithm predicting the probability $\hat{\sigma}_{i\alpha}$ that a replica node (i,α) is in the RMCGC when the initial damage $\{s_i\}$ is randomly drawn for the probability distribution $\hat{\mathcal{P}}(\{s_{i\alpha}\})$. Assuming that each replica node is damaged independently the probability distribution $\hat{\mathcal{P}}(\{s_{i\alpha}\})$ is given by

$$\hat{\mathcal{P}}(\{s_{i\alpha}\})=\prod_{i=1}^{N}\prod_{\alpha=1}^{M}p^{s_{i\alpha}}(1-p)^{1-s_{i\alpha}}. \tag{SM11}$$

The message passing algorithm valid for a random distribution of the initial disorder, is written in terms of the messages $\hat{\sigma}_{i\to j}^{\vec{m},\vec{n}}(\vec{s})$. The messages $\hat{\sigma}_{i\to j}^{\vec{m},\vec{n}}(\vec{s})$ take real values between zero and one. They indicate the probability that node i send to node j a message $\vec{n}=\vec{n}_{i\to j}$ given that node j is connected to node i by a multilink $\vec{m}=\vec{m}_{ij}$ and that node j has initial damage configuration $\vec{s}=\vec{s}_j$, i.e. $(s_1,s_2,\ldots,s_\alpha,\ldots,s_M)=(s_{j1},s_{j2},\ldots,s_{jM})$.

Let us indicate with $\hat{P}(\vec{s})$ the probability of a local initial damage configuration given by

$$\hat{P}(\vec{s})=\prod_{\alpha=1}^{M}p^{s_\alpha}(1-p)^{1-s_\alpha} \tag{SM12}$$

and let us indicate with $\vec{r}$ the vector

$$\vec{r}=(r^{[1]},r^{[2]},\ldots,r^{[\alpha]},\ldots,r^{[M]}) \tag{SM13}$$

of elements $r^{[\alpha]}=0,1$. Using this notation, the messages $\hat{\sigma}_{i\to j}^{\vec{m},\vec{n}}(\vec{s})$ are determined by the following algorithm (see last section of this Supplementary Information for the derivation of these results):

(a) if $\nu=\sum_{\alpha=1}^M n^{[\alpha]}>1$ and $\vec{m}=\vec{m}_{ij}$,

$$\begin{aligned}\hat{\sigma}_{i\to j}^{\vec{m},\vec{n}}(\vec{s})\;&=\;\sum_{\vec{s}_i|\sum_\alpha s_{i\alpha}>1}\hat{P}(\vec{s}_i)\sum_{\vec{r}|r^{[\alpha]}=0\text{ if }\left(n^{[\alpha]}+(1-n^{[\alpha]})m^{[\alpha]}s_\alpha\right)=0}\mathcal{C}^{\vec{m},\vec{n}}(\vec{s}_i,\vec{s},\vec{r})\\&\times\left[\prod_{\ell\in N(i)\setminus j}\left(1-\sum_{\vec{n}'|\sum_\alpha(n')^{[\alpha]}r^{[\alpha]}>0}\hat{\sigma}_{\ell\to i}^{\vec{m}_{\ell i}\vec{n}'}(\vec{s}_i)\right)\right],\end{aligned} \tag{SM14}$$

where

$$\mathcal{C}^{\vec{m},\vec{n}}(\vec{s}_i,\vec{s},\vec{r})=\prod_{\alpha=1}^{M}\left[(m^{[\alpha]}s_{i\alpha}s_\alpha)^{n^{[\alpha]}}(-1)^{r^{[\alpha]}n^{[\alpha]}}(1-s_{i\alpha})^{(1-r^{[\alpha]})(1-n^{[\alpha]})m^{[\alpha]}s_\alpha}\,(s_{i\alpha})^{r^{[\alpha]}(1-n^{[\alpha]})m^{[\alpha]}s_\alpha}\right], \tag{SM15}$$

(b) if $\nu=\sum_{\alpha'=1}^{M}n^{[\alpha']}=1$, $n^{[\alpha]}=1$ and $\vec{m}=\vec{m}_{ij}$,

$$\begin{aligned}\hat{\sigma}_{i\to j}^{\vec{m},\vec{n}}(\vec{s}) \;=\; & \sum_{\vec{s}_i|\sum_{\alpha'}s_{i\alpha'}>1}\hat{P}(\vec{s}_i)s_{i\alpha}s_\alpha a_{ij}^{[\alpha]}\left\{1-\prod_{\ell\in N(i)\setminus j}\left(1-\sum_{\vec{n}'|(n')^{[\alpha]}>0}\hat{\sigma}_{\ell\to i}^{\vec{m}_{\ell i}\vec{n}'}(\vec{s}_i)\right)\right.\\ & -\sum_{\vec{r}|r^{[\alpha]}=0}\prod_{\alpha'\neq\alpha}(1-s_{i\alpha'})^{\left(1-r^{[\alpha']}\right)}(s_{i\alpha'})^{r^{[\alpha']}}\prod_{\ell\in N(i)\setminus j}\left(1-\sum_{\vec{n}'|\sum_{\alpha'}(n')^{[\alpha']}r^{[\alpha']}>0}\hat{\sigma}_{\ell\to i}^{\vec{m}_{\ell i}\vec{n}'}(\vec{s}_i)\right)\\ & \left.+\sum_{\vec{r}|r^{[\alpha]}=0}\prod_{\alpha'\neq\alpha}(1-s_{i\alpha'})^{\left(1-r^{[\alpha']}\right)}(s_{i\alpha'})^{r^{[\alpha']}}\prod_{\ell\in N(i)\setminus j}\left(1-\sum_{\vec{n}'|\sum_{\alpha'}(n')^{[\alpha']}\left[\delta_{\alpha,\alpha'}+r^{[\alpha']}\right]>0}\hat{\sigma}_{\ell\to i}^{\vec{m}_{\ell i}\vec{n}'}(\vec{s}_i)\right)\right\}\end{aligned} \tag{SM16}$$

(c) if $\nu=\sum_{\alpha=1}^{M}n^{[\alpha]}=0$ and $\vec{m}=\vec{m}_{ij}$,

$$\hat{\sigma}_{i\to j}^{\vec{m},\vec{0}}(\vec{s})=1-\sum_{\vec{n}\neq\vec{0}}\hat{\sigma}_{i\to j}^{\vec{m},\vec{n}}(\vec{s}). \tag{SM17}$$

Finally the probability $\hat{\sigma}_{i\alpha}$ that a replica node (i,α) is in the RMCGC is given by

$$\begin{aligned}\hat{\sigma}_{i\alpha} \;=\; & \sum_{\vec{s}_i|\sum_{\alpha'}s_{i\alpha'}>1}\hat{P}(\vec{s}_i)s_{i\alpha}a_{ij}^{[\alpha]}\left\{1-\prod_{\ell\in N(i)}\left(1-\sum_{\vec{n}'|(n')^{[\alpha]}>0}\hat{\sigma}_{\ell\to i}^{\vec{m}_{\ell i}\vec{n}'}(\vec{s}_i)\right)\right.\\ & -\sum_{\vec{r}|r^{[\alpha]}=0}\prod_{\alpha'\neq\alpha}(1-s_{i\alpha'})^{\left(1-r^{[\alpha']}\right)}(s_{i\alpha'})^{r^{[\alpha']}}\prod_{\ell\in N(i)}\left(1-\sum_{\vec{n}'|\sum_{\alpha'}(n')^{[\alpha']}r^{[\alpha']}>0}\hat{\sigma}_{\ell\to i}^{\vec{m}_{\ell i}\vec{n}'}(\vec{s}_i)\right)\\ & \left.+\sum_{\vec{r}|r^{[\alpha]}=0}\prod_{\alpha'\neq\alpha}(1-s_{i\alpha'})^{\left(1-r^{[\alpha']}\right)}(s_{i\alpha'})^{r^{[\alpha']}}\prod_{\ell\in N(i)}\left(1-\sum_{\vec{n}'|\sum_{\alpha'}(n')^{[\alpha']}\left[\delta_{\alpha,\alpha'}+r^{[\alpha']}\right]>0}\hat{\sigma}_{\ell\to i}^{\vec{m}_{\ell i}\vec{n}'}(\vec{s}_i)\right)\right\}.\end{aligned} \tag{SM18}$$

Average over multilayer ensemble with give multidegree sequence

In order to derive the phase diagram of the percolation transition in presence of redundant interdependencies over given multilayer network ensembles, it is useful to consider a further average of the messages $\hat{\sigma}_{i\to j}^{\vec{m},\vec{n}}$. To this end we consider the multilayer network ensemble that preserves the multidegree sequence $\{k_i^{\vec{m}}\}$. Every multilayer network $\vec{G}$ in this ensemble has probability

$$\tilde{\mathcal{P}}(\vec{G})=\frac{1}{\tilde{Z}}\prod_{i=1}^{N}\prod_{\vec{m}\neq\vec{0}}\delta\left[k_i^{\vec{m}},\sum_{j=1}^{N}\delta\left(\vec{m},\vec{m}_{ij}\right)\right], \tag{SM19}$$

where $\tilde{Z}$ is a normalization constant equal to the number of multilayer networks with the given multidegree sequence.

In this multilayer network ensemble the average messages $S^{\vec{m},\vec{n}}(\vec{s})=\left\langle\hat{\sigma}_{i\to j}^{\vec{m}_{ij},\vec{n}}|\vec{m}=\vec{m}_{ij}\right\rangle$ indicate the probability that a message $\vec{n}$ is sent toward a node with initial damage configuration $\vec{s}$ over a multilink $\vec{m}$. These average messages can be found by solving the following recursive equations:

(a) if $\nu=\sum_{\alpha=1}^{M} n^{[\alpha]}>1$

$$
\begin{aligned}
S^{\vec{m},\vec{n}}(\vec{s}) \; &= \; \sum_{\{k^{\vec{m}}\}} \frac{k^{\vec{m}}}{\langle k^{\vec{m}}\rangle} P(\{k^{\vec{m}}\}) \sum_{\vec{s}_i\,|\,\sum_\alpha s_{i\alpha}>1} \hat{P}(\vec{s}_i) \sum_{\vec{r}|r^{[\alpha]}=0 \text{ if } \left(n^{[\alpha]}+(1-n^{[\alpha]})m^{[\alpha]}s_\alpha\right)=0} \mathcal{C}^{\vec{m},\vec{n}}(\vec{s}_i,\vec{s},\vec{r}) \\
&\times \left[\prod_{\vec{m}'\neq 0}\left(1-\sum_{\vec{n}'\,|\,\sum_\alpha (n')^{[\alpha]}r^{[\alpha]}>0} S^{\vec{m}'\vec{n}'}(\vec{s}_i)\right)^{k^{\vec{m}'}-\delta(\vec{m},\vec{m}')}\right],
\end{aligned} \tag{SM20}
$$

where

$$
\mathcal{C}^{\vec{m},\vec{n}}(\vec{s}_i,\vec{s},\vec{r})=\prod_{\alpha=1}^{M}\left[(m^{[\alpha]}s_{i\alpha}s_\alpha)^{n^{[\alpha]}}(-1)^{r^{[\alpha]}n^{[\alpha]}}(1-s_{i\alpha})^{(1-r^{[\alpha]})(1-n^{[\alpha]})m^{[\alpha]}s_\alpha}\,(s_{i\alpha})^{r^{[\alpha]}(1-n^{[\alpha]})m^{[\alpha]}s_\alpha}\right], \tag{SM21}
$$

(b) if $\nu=\sum_{\alpha'=1}^{M} n^{[\alpha']}=1$, $n^{[\alpha]}=1$

$$
\begin{aligned}
S^{\vec{m},\vec{n}}(\vec{s}) \; &= \; \sum_{\{k^{\vec{m}}\}} \frac{k^{\vec{m}}}{\langle k^{\vec{m}}\rangle} P(\{k^{\vec{m}}\}) \sum_{\vec{s}_i\,|\,\sum_{\alpha'} s_{i\alpha'}>1} \hat{P}(\vec{s}_i)s_{i\alpha}s_\alpha a_{ij}^{[\alpha]} \left\{1-\prod_{\vec{m}'\neq\vec{0}}\left(1-\sum_{\vec{n}'|(n')^{[\alpha]}>0} S^{\vec{m}'\vec{n}'}(\vec{s}_i)\right)^{k^{\vec{m}'}-\delta(\vec{m},\vec{m}')}\right. \\
&- \sum_{\vec{r}|r^{[\alpha]}=0}\prod_{\alpha'\neq\alpha}(1-s_{i\alpha'})^{\left(1-r^{[\alpha']}\right)}(s_{i\alpha'})^{r^{[\alpha']}}\prod_{\vec{m}'\neq\vec{0}}\left(1-\sum_{\vec{n}'\,|\,\sum_{\alpha'}(n')^{[\alpha']}r^{[\alpha']}>0} S^{\vec{m}'\vec{n}'}(\vec{s}_i)\right)^{k^{\vec{m}}-\delta(\vec{m},\vec{m}')} \\
&\left. + \sum_{\vec{r}|r^{[\alpha]}=0}\prod_{\alpha'\neq\alpha}(1-s_{i\alpha'})^{\left(1-r^{[\alpha']}\right)}(s_{i\alpha'})^{r^{[\alpha']}}\prod_{\vec{m}'\neq\vec{0}}\left(1-\sum_{\vec{n}'\,|\,\sum_{\alpha'}(n')^{[\alpha']}\left[\delta_{\alpha,\alpha'}+r^{[\alpha']}\right]>0} S^{\vec{m}'\vec{n}'}(\vec{s}_i)\right)^{k^{\vec{m}}-\delta(\vec{m},\vec{m}')}\right\},
\end{aligned} \tag{SM22}
$$

(c) if $\nu=\sum_{\alpha=1}^{M} n^{[\alpha]}=0$

$$
S^{\vec{m},\vec{0}}(\vec{s})=1-\sum_{\vec{n}\neq\vec{0}} S^{\vec{m},\vec{n}}(\vec{s}). \tag{SM23}
$$

Finally the probability S_α that a replica node in layer α is in the RMCGC in the multilayer network ensemble is given by

$$
\begin{aligned}
S_\alpha \; &= \; \sum_{\{k^{\vec{m}}\}} P(\{k^{\vec{m}}\}) \sum_{\vec{s}_i\,|\,\sum_{\alpha'} s_{i\alpha'}>1} \hat{P}(\vec{s}_i)s_{i\alpha} a_{ij}^{[\alpha]} \left\{1-\prod_{\vec{m}'\neq\vec{0}}\left(1-\sum_{\vec{n}'|(n')^{[\alpha]}>0} S^{\vec{m}'\vec{n}'}(\vec{s}_i)\right)^{k^{\vec{m}'}}\right. \\
&- \sum_{\vec{r}|r^{[\alpha]}=0}\prod_{\alpha'\neq\alpha}(1-s_{i\alpha'})^{\left(1-r^{[\alpha']}\right)}(s_{i\alpha'})^{r^{[\alpha']}}\prod_{\vec{m}'\neq\vec{0}}\left(1-\sum_{\vec{n}'\,|\,\sum_{\alpha'}(n')^{[\alpha']}r^{[\alpha']}>0} S^{\vec{m}'\vec{n}'}(\vec{s}_i)\right)^{k^{\vec{m}}} \\
&\left. + \sum_{\vec{r}|r^{[\alpha]}=0}\prod_{\alpha'\neq\alpha}(1-s_{i\alpha'})^{\left(1-r^{[\alpha']}\right)}(s_{i\alpha'})^{r^{[\alpha']}}\prod_{\vec{m}'\neq\vec{0}}\left(1-\sum_{\vec{n}'\,|\,\sum_{\alpha'}(n')^{[\alpha']}\left[\delta_{\alpha,\alpha'}+r^{[\alpha']}\right]>0} S^{\vec{m}'\vec{n}'}(\vec{s}_i)\right)^{k^{\vec{m}}}\right\}
\end{aligned} \tag{SM24}
$$

Derivation of Eq. $(SM14)$

In this section, we will discuss in detail the derivation of Eq. $(SM14)$ from Eq.$(SM6)$. A similar derivation (that we omit here) can be performed to derive Eqs. $(SM16)/(SM18)$ from Eqs. $(SM7)/(SM10)$.

We start from Eq. (SM6) written for the messages $\sigma_{i\to j}^{\vec{m},\vec{n}}$ sent from a node i to a node j with $\vec{n}$ satisfying $\nu = \sum_{\alpha=1}^{M} n^{[\alpha]} > 1$ and $\vec{m} = (a_{ij}^{[1]}, a_{ij}^{[2]} \ldots, a_{ij}^{[M]})$. This equation is rewritten here for convenience,

$$\sigma_{i\to j}^{\vec{m},\vec{n}} = \prod_{\alpha=1}^{M}\left[m^{[\alpha]}s_{j\alpha}s_{i\alpha} - m^{[\alpha]}s_{j\alpha}s_{i\alpha}\prod_{\ell\in N(i)\setminus j}\left(1-n_{\ell\to i}^{[\alpha]}\right)\right]^{n^{[\alpha]}}\prod_{\alpha=1}^{M}\left[1-s_{i\alpha}+s_{i\alpha}\prod_{\ell\in N(i)\setminus j}\left(1-n_{\ell\to i}^{[\alpha]}\right)\right]^{(1-n^{[\alpha]})m^{[\alpha]}s_{j\alpha}} \tag{SM25}$$

We given a set of variables $p^{[\alpha]} = 0, 1$ we can use the following identity

$$\prod_{\alpha=1}^{M}(y_\alpha+z_\alpha)^{p^{[\alpha]}} = \prod_{\alpha|p^{[\alpha]}>0}(y_\alpha+z_\alpha) = \sum_{\vec{r}|r^{[\alpha]}=0 \text{ if } p^{[\alpha]}=0}\prod_{\alpha=1}^{M}\left[(y_\alpha)^{1-r^{[\alpha]}}(z_\alpha)^{r^{[\alpha]}}\right], \tag{SM26}$$

where in the last expression we perform a sum over all the M-dimensional vectors $\vec{r}$

$$\vec{r} = (r^{[1]}, r^{[2]}, \ldots, r^{[\alpha]}, \ldots, r^{[M]}), \tag{SM27}$$

with $r^{[\alpha]} = 0, 1$ if $p^{[\alpha]} = 1$ and $r^{[\alpha]} = 0$ if $p^{[\alpha]} = 0$. Using this expansion for the products in Eq. $(SM25)$ we obtain

$$\sigma_{i\to j}^{\vec{m},\vec{n}} = \sum_{\vec{r}|r^{[\alpha]}=0 \text{ if } \left(n^{[\alpha]}+(1-n^{[\alpha]})m^{[\alpha]}s_{j\alpha}\right)=0}\mathcal{C}^{\vec{m},\vec{n}}(\vec{s}_i,\vec{s},\vec{r})\prod_{\ell\in N(i)\setminus j}\left[\prod_{\alpha=1}^{M}\left(1-n_{\ell\to i}^{[\alpha]}\right)^{r^{[\alpha]}}\right], \tag{SM28}$$

where $\mathcal{C}^{\vec{m},\vec{n}}(\vec{s}_i,\vec{s},\vec{r})$ is given by Eq. $(SM15)$. By using the fact that the messages $\sigma_{i\to j}^{\vec{m},\vec{n}}$ take only values zero or one, that that out of all the messages $\sigma_{i\to j}^{\vec{m},\vec{n}}$ from node i to node j only one is actually equal to one, and all the others are zero, we can rewrite Eq. $(SM28)$ as

$$\sigma_{i\to j}^{\vec{m},\vec{n}} = \sum_{\vec{r}|r^{[\alpha]}=0 \text{ if } \left(n^{[\alpha]}+(1-n^{[\alpha]})m^{[\alpha]}s_{j\alpha}\right)=0}\mathcal{C}^{\vec{m},\vec{n}}(\vec{s}_i,\vec{s},\vec{r})\prod_{\ell\in N(i)\setminus j}\left(1-\sum_{\vec{n}'|\sum_\alpha (n')^{[\alpha]}r^{[\alpha]}>0}\sigma_{\ell\to i}^{\vec{m}_{\ell i}\vec{n}'}\right). \tag{SM29}$$

Finally, averaging over the probability distribution $\hat{P}(\vec{s}_i)$ of the configuration $\vec{s}_i$ of the initial damage of node i, in the locally treelike approximation we obtain for the messages $\hat{\sigma}_{i\to j}^{\vec{m},\vec{n}}(\vec{s}_j)$ the Eq. $(SM14)$ that we rewrite here for convenience,

$$\begin{aligned}\hat{\sigma}_{i\to j}^{\vec{m},\vec{n}}(\vec{s}_j) = &\sum_{\vec{s}_i|\sum_\alpha s_{i\alpha}>1}P(\vec{s}_i)\sum_{\vec{r}|r^{[\alpha]}=0 \text{ if } \left(n^{[\alpha]}+(1-n^{[\alpha]})m^{[\alpha]}s_{j\alpha}\right)=0}\mathcal{C}^{\vec{m},\vec{n}}(\vec{s}_i,\vec{s},\vec{r})\\ &\prod_{\ell\in N(i)\setminus j}\left(1-\sum_{\vec{n}'|\sum_\alpha (n')^{[\alpha]}r^{[\alpha]}>0}\hat{\sigma}_{\ell\to i}^{\vec{m}_{\ell i}\vec{n}'}(\vec{s}_i)\right).\end{aligned} \tag{SM30}$$