Communities are fundamental entities for the characterization of the structure of real networks. The standard approach to the identification of communities in networks is based on the optimization of a quality function known as modularity. Although modularity has been at the center of an intense research activity and many methods for its maximization have been proposed, not much is yet known about the necessary conditions that communities need to satisfy in order to be detectable with modularity maximization methods. Here, we develop a simple theory to establish these conditions, and we successfully apply it to various classes of network models. Our main result is that heterogeneity in the degree distribution helps modularity to correctly recover the community structure of a network and that, in the realistic case of scale-free networks with degree exponent $\gamma < 2.5$, modularity is always able to detect the presence of communities.
and
\[ A_{2,2}[v_2] + A_{2,1}[v_1] - \frac{\langle s_1|v_1 \rangle + \langle s_2|v_2 \rangle}{\langle s_1|1 \rangle + \langle s_2|1 \rangle} s_2 = \lambda|v_2\rangle. \] (3)

If we multiply them for (1), and then take their difference, we obtain
\[
\lambda(|v_1|1) - (v_2|1) = ((s_1|v_1) - (s_2|v_2) + (s_2|v_2) - (s_1|v_1)) - \alpha(s_1|v_1) - (s_2|v_2)),\]
where we have defined \( \alpha = \frac{|v_1|1}{|v_1|1 + |v_2|1} \). For simplicity, in the following we will consider only cases in which \( \alpha \approx 0 \), i.e., cases in which both groups have a comparable total number of edges.

We define the detectability regime as the regime in which the two preimposed communities can be detected by means of modularity spectral optimization. This regime is characterized by the fact that the components of the principal eigenvector corresponding to the nodes of one of the modules have coherent signs, while the two portions of the eigenvector corresponding to different groups are opposite in sign [10]. If we suppose that these modules are uncorrelated graphs (i.e., without further internal subcommunity structure) with prescribed in- and out-strength vectors, we expect that this eigenvector is such that
\[
|v_1| = n_1 (|s_{1,1}| - |s_{2,1}|) = n_1 |\Delta s_1| \]
and
\[
|v_2| = n_2 (|s_{2,2}| - |s_{1,2}|) = n_2 |\Delta s_2|,\]
where \( n_1 \) and \( n_2 \) are proportionality constants, while \( |\Delta s_1| \) and \( |\Delta s_2| \) are, respectively, the vectors whose entries are given by the difference of the in and out strengths of the nodes in the groups 1 and 2. Equations (5) and (6) simply state that the coordinates of \( |v_1| \) and \( |v_2| \) are linearly proportional to the difference of the in- and out-strength vectors, a solution that appears natural if we interpret the modularity function as the stationary solution of a random walk between the two communities [16–18]. This conjecture is indeed perfectly verified in numerical estimations of the largest eigenvector of the modularity matrix (see Fig. 1 and [19]), and thus Eqs. (5) and (6) can be used as a reasonable ansatz for the solution of our problem. If we finally insert Eqs. (5) and (6) into Eq. (4), we can give an estimate of the largest eigenvalue \( \lambda_{\text{max}} \) of the modularity matrix in the regime of detectable communities, and write
\[
\lambda_{\text{max}} = \frac{1}{2} \left( \frac{|\Delta s_1| |\Delta s_1|}{|s_1|1} + \frac{|\Delta s_2| |\Delta s_2|}{|s_2|1} \right),\]
where we have used the orthogonality condition \( \langle v_1|1 \rangle + \langle v_2|1 \rangle = 0 \), which leads to \( n_1 = -n_2 (|\Delta s_2| |s_1|1)/|\Delta s_1|1 \). Note that expression (7) is valid for given strength vectors. If we instead assume that the entries of these vectors are random variates obeying the statistical distributions \( P(|s_1|1) \) and \( P(|s_2|1) \), we can write
\[
\lambda_{\text{max}} = \frac{1}{2} \left( \frac{m_1^{(1)}}{m_1^{(1)}} + \frac{m_2^{(2)}}{m_2^{(2)}} \right),\]
where \( m_1^{(1)} \) and \( m_2^{(2)} \) are, respectively, the first and the second moments of the distribution \( P(|s_i|1) \).

It is important to stress that Eqs. (7) and (8) give us an estimate of the largest eigenvalue of \( Q \) only in the detectability regime. If the structure of the entire graph is instead such that the two modules are not detectable by means of modularity maximization, there will be another principal eigenvector orthogonal to the previous one, and thus not showing the presence of the two modules. Since we have supposed that both modules are randomly generated graphs, the other eigenvalue that is competing with \( \lambda_{\text{max}} \) for being the highest eigenvalue of the modularity matrix is given by the second largest eigenvalue of the annealed random network associated to \( Q \) [14]. In intuitive terms, this means that, in the regime in which the groups are undetectable, the signal present in \( A \) is not sufficiently high, and \( Q \) is in spectral terms indistinguishable from \( P \). In the following, we will consider some examples of network ensembles where both these eigenvalues can be analytically estimated, and thus the detectability problem can be explicitly solved.

**Regular graphs.** In this case, each node has exactly \( c_{\text{in}} \) random connections with other nodes in its group, and \( c_{\text{out}} \) random connections outside its own group. Equation (4) reduces to
\[
(\lambda_{\text{max}} - c_{\text{in}} + c_{\text{out}})|v_1|1 = 0,
\]
thus either (i) \( |v_1|1 = |v_2|1 = 0 \) and \( \lambda_{\text{max}} \neq c_{\text{in}} - c_{\text{out}} \) or (ii) \( \lambda_{\text{max}} = c_{\text{in}} - c_{\text{out}} \), \( |v_1|1 \neq 0 \), and \( |v_2|1 \neq 0 \). In case...
matrix is given by the second largest eigenvalue of a random graph with average degree \( c = c_{\text{in}} + c_{\text{out}} \), that is \( 2\sqrt{c} \) [14,23], and this finally leads to the detectability threshold

\[
c_{\text{in}} - c_{\text{out}} = \sqrt{c_{\text{in}} + c_{\text{out}}},
\]

as already obtained in [14,24]. The results of numerical simulations perfectly agree with our theoretical prediction (see Fig. 2). The prediction appears to be not visibly dependent on the system size, and already for small networks Eq. (12) represents a very good estimate of the transition point. We note that, as in the case of regular graphs, for a large portion of the region \( c_{\text{in}} > c_{\text{out}} \), modularity fails to recover the community structure of the graph. It is, however, interesting to stress that the detectability threshold is two times smaller than the one registered for regular graphs, and thus the heterogeneity in node degrees seems to enhance the ability of modularity to detect communities.

\textbf{LFR benchmark graphs.} As a final example, we consider a special case of the benchmark graphs introduced by Lancichinetti et al. [25]. We set \( |s_{1,1}| = |s_{2,1}| = c_{\text{in}}|s| \) and \( |s_{1,2}| = |s_{2,1}| = c_{\text{out}}|s| \), where the entries of the vector \([s]\) are random variates in the range 1 to \( N \) taken from a power-law distribution with exponent \( \gamma \). Equation (8) becomes

\[
\lambda_{\text{max}} = (c_{\text{in}} - c_{\text{out}}) \frac{\zeta_{N}(\gamma - 2)}{\zeta_{N}(\gamma - 1)},
\]

where \( \zeta_{N}(x) = \sum_{n=1}^{N} n^{-x} \) is the Riemann zeta function truncated at the \( N \)th term. The term of comparison for the largest eigenvalue of the modularity matrix is still given by the second largest eigenvalue of the annealed random graph associated with the modularity matrix. We do not have an exact guess on how this quantity depends on parameters of the network model, but we can use the upper bound of the largest eigenvalue of random scale-free graphs to get more insights. According to the predictions by Chung et al. adapted to the present case, the largest eigenvalue \( \mu_{\text{max}} \) of our random scale-free graphs is equal to the maximum between \( \mu_{\text{max}}^{(1)} = (c_{\text{in}} + c_{\text{out}}) \left( \frac{c_{\text{in}} + c_{\text{out}}}{\sqrt{c_{\text{in}} + c_{\text{out}}}} \right)^{\gamma - 1} \) and \( \mu_{\text{max}}^{(2)} = \sqrt{c_{\text{in}} + c_{\text{out}}}/\max \), each \( s_{\text{max}} \) the largest degree in the network [23,26]. In the limit of sufficiently large \( N \), we have that for \( \gamma < 2.5 \), the dominating eigenvalue is \( \mu_{\text{max}}^{(1)} \); for \( \gamma > 2.5 \), the largest eigenvalue is instead \( \mu_{\text{max}}^{(2)} \). This has very important implications when compared to our prediction of \( \lambda_{\text{max}} \) given in (13): (i) For \( \gamma < 2.5 \), \( \lambda_{\text{max}} \) grows as fast as \( \mu_{\text{max}} \) with the system size, thus the detectability threshold should approach zero as \( N \) increases. (ii) For \( \gamma > 2.5 \), \( \mu_{\text{max}} \) grows faster than \( \lambda_{\text{max}} \) as \( N \) increases. The detectability threshold should grow with the system size, and eventually converge to a finite fixed value (for instance, for \( \gamma \to \infty \) we must recover the result valid for the case of regular graphs).

The results of numerical simulations support our thesis (see Fig. 3). When we plot \( \lambda_{\text{max}}^{(2)} \) as a function of \( c_{\text{in}} - c_{\text{out}} \), with \( q_{1} \) and \( q_{2} \), respectively, the first and second moments of the strength distribution of the network, we see that when \( \gamma < 2.5 \) this quantity slowly approaches, as \( N \) increases, the linear behavior \( c_{\text{in}} - c_{\text{out}} \) as predicted by Eq. (13). This means that, in the limit of infinite large systems, modularity is able to detect the presence of the network blocks for every \( c_{\text{in}} > c_{\text{out}} \).
Although the qualitative outcome does not depend on this choice. From functions of\(c\) as a function of \(\gamma\) is smaller than the sum of all strengths (i.e., \(\sum c_i\)). The results presented here have been obtained by averaging over 100 different realizations of the models. To suppress fluctuations, we restricted to the case of networks for which the square of the largest strength is smaller than the sum of all strengths (i.e., \(\sum c_i < \sum s_i\)) [23,26], although the qualitative outcome does not depend on this choice. For clarity, we placed arrows in the various panels to indicate the direction of increasing \(N\). (a), (c) Largest eigenvalue \(\lambda_{\text{max}}\) multiplied by \(q_i/q_s\) as a function of \(c_{\text{in}} - c_{\text{out}}\). The full black line corresponds to \(c_{\text{in}} - c_{\text{out}}\) as functions of \(c_{\text{in}} - c_{\text{out}}\).

Instead, for \(\gamma > 2.5\), the lower part of the curve tends to move away from the linear behavior as the system size grows. This implies that there will be also in the limit of infinitely large systems, always a part of the \(c_{\text{in}} > c_{\text{out}}\) region in which the two blocks are undetectable via modularity maximization.

To summarize, we identified the necessary conditions that communities need to satisfy in order to be detectable by means of modularity maximization. Our results are valid for the case of two groups with a comparable number of edges, and when the information about the number of such groups is used as ingredient in the maximization of the modularity function. Our main result is that in random network ensembles with preimposed community structure, the eigenvector of the modularity matrix that identifies the presence of the block structure is associated with an eigenvalue approximately equal to the ratio between the second and the first moments of the distribution of the difference between internal and external node strengths. If this eigenvalue is larger than the second largest eigenvalue of the null model associated to the modularity function, then modularity is able to detect such a structure, otherwise not. This represents a limitation in the case of graphs with homogeneous degrees. Increasing the heterogeneity of the network accelerates instead the ability of modularity to recover the correct community structure. For example, adding noise to a regular graph makes the detectability threshold two times smaller. More importantly, if the heterogeneity of the node degrees is sufficiently high, as in the case of real networked systems, then modularity is always able to detect communities.

The author thanks A. Arenas, R. Darst, and S. Fortunato for helpful discussions on the subject of this article. The author acknowledges support from the Spanish Ministerio de Ciencia e Innovación through the Ramón y Cajal program.

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