On Categorical Models of GoI

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In this lecture

- We shall talk about the first categorical model of GoI.
- We will consider GoI 1 (Girard 1989) for MELL.
- I shall follow the paper: Haghverdi & Scott, *A Categorical Model for GoI*, ICALP 2004 and TCS 2006.
- We emphasize the notion of categorical *trace*.

Sense/Denotation

A critique of reductionism

G. Frege (1848-1925): In *Function und Begriff*, 1891.

- Sinn/Bedeutung
  sense/denotation
- The sense constitutes the particular way in which its denotation (reference) is given to one who grasps the thought.
- $2 + 3 = 5$
- sense/denotation
  dynamic/static

Example

$$\frac{A \vdash A \quad A \vdash A}{A \vdash A} \quad \succ \quad A \vdash A$$

- $id_A \circ id_A = id_A$
- More generally, Π, Π' proofs of $\Gamma \vdash A$, $\Pi \succ \Pi'$.
- Then
$$[\![\Pi]\!] = [\![\Pi']\!] : [\![\Gamma]\!] \longrightarrow [\![A]\!].$$
- A *static* view!
- GoI offers a dynamic semantics.
- Syntax carries irrelevant information.

Dynamics

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- Theorem (Cut Elimination (Hauptsatz))

  (Gentzen, 1934)
  If Π is a proof of a sequent $\Gamma \vdash A$, then there is a proof Π' of the same sequent which does not use the cut rule.

$$\frac{\Gamma \vdash A \quad A, \Delta \vdash B}{\Gamma, \Delta \vdash B}\ (cut\ rule)$$

Girard's Implementation (System $\mathcal{F}$)

Π	$\rightsquigarrow$	(u, σ)
a proof of second order LL (no additives)		a pair of partial symmetries in $\mathbb{B}(\mathcal{H})$

- Dynamics = elimination of cuts (σ) using

$$EX(u, \sigma) = (1 - \sigma^2) \sum_{n \geq 0} u(\sigma u)^n (1 - \sigma^2)$$

- Theorem (Girard, 1987)

  (i) If (u, σ) is the interpretation of a proof Π of a sequent $\vdash [\Delta], \Gamma$ then σu is nilpotent.
  (ii) if Γ does not use the symbols "?" or "$\exists$", then the interpretation is sound.

  - strong normalisation $\leftrightarrow$ nilpotency

Back to our example

$$\frac{\vdash A, A^{\perp} \quad \vdash A, A^{\perp}}{\vdash [A^{\perp}, A], A, A^{\perp}} \qquad \succ \qquad \vdash A, A^{\perp}$$

- proofs as matrices on $\mathcal{M}_{2m+n}(\mathbb{B}(\ell^2))$
- $u = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \qquad \sigma = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
- Dynamics: $EX(u, \sigma) = (1 - \sigma^2)(u + u\sigma u)(1 - \sigma^2) =$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

The big picture

$$\begin{array}{ccc} & \text{proof} & \rightsquigarrow & \text{algorithm} \\ \text{cut-elim.} & \downarrow & & \downarrow & \text{computation} \\ & \text{cut-free proof} & \rightsquigarrow & \text{datum} \end{array}$$

$$\begin{array}{ccc} & \Pi & \rightsquigarrow & \theta(\Pi) \\ \text{cut-elim.} & \downarrow & & \downarrow & \text{computation} \\ & \Pi' & \rightsquigarrow & \theta(\Pi') = EX(\theta(\Pi), \sigma) \end{array}$$

Wish list!

- Connect to other areas in mathematics to draw on their techniques:
  - Knot Theory: view proofs as knots, use knot invariants.
  - Algebraic Topology: view proofs as topological spaces, use homology.

  Examples abound: Mulmuley (Alg. Geometry and Geometric Invariant Theory)
  J. Friedman: Grothendieck topology
  Combinatorics,...
- Study logical complexity using GoI tools.
  - PTIME, LOGSPACE.
  - BLL (GSS), ELL and LLL (Girard).

A Brief History, *with apologies*

- GoI 2 (1988): Deadlock-free algorithms, Recursion
- GoI 3 (1995): Additives
- GoI 4 (2003): The feedback equation
- GoI 5 (2008): The hyperfinite factor
- Danos (1990): Untyped Lambda Calculus
- Danos, Regnier, Malacaria, Mackie : Path-based Semantics
- Logical complexity related work, optimal lambda reduction, etc

History, cont'd

- Abramsky & Jagadeesan (1994): Categorical interpretation using Domain Theory, Feedback in dataflow networks
- Abramsky (1997): GoI Situation, Abramsky's Program
- Haghverdi (PhD, 2000): UDC based (particle style) GoI Situation and more, including path-based semantics
- Abramsky, Haghverdi and Scott (2002): GoI Situation to CA
- Haghverdi, Scott (2004,2006): Categorical models
- Haghverdi, Scott (2005,2009): Typed GoI
- Hines (1997): Self-similarity, inverse semigroups

Σ-Monoids

Definition (Kuros,Higgs,Manes,Arbib,Benson)

(M, Σ), where M is a nonempty set and Σ is a partial operation on countable families in M. $\{x_i\}_{i \in I}$ is *summable* if $\Sigma_{i \in I} x_i$ is defined subject to:

- *Partition-Associativity:* $\{x_i\}_{i \in I}$ and $\{I_j\}_{j \in J}$ a countable partition of I
$$\Sigma_{i \in I} x_i = \Sigma_{j \in J}(\Sigma_{i \in I_j} x_i).$$
- *Unary sum:* $\Sigma_{i \in \{j\}} x_i = x_j$.

Facts about Σ-Monoids

- $\Sigma_{i \in \emptyset} x_i$ exists and is denoted by 0. It is a countable additive identity.
- Sum is commutative and associative whenever defined.
- $\Sigma_{i \in I} x_{\varphi(i)}$ is defined for any permutation φ of I, whenever $\Sigma_{i \in I} x_i$ exits.
- There are **no** additive inverses: $x + y = 0$ implies $x = y = 0$.

Examples

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- $(\Sigma_I f_i)(x)$ as above.

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- $M =$ countably complete poset, $\Sigma = sup$.

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- $x + (y + z)$ is defined but $(x + y) + z$ is not defined.

Unique Decomposition Categories (UDCs)

Definition

A *unique decomposition category* $\mathbb{C}$ is a symmetric monoidal category where:

- Every homset is a Σ-Monoid
- Composition distributes over sum (careful!)

satisfying the axiom:

(A) For all $j \in I$

- *quasi injection*: $\iota_j : X_j \longrightarrow \otimes_I X_i$,
- *quasi projection*: $\rho_j : \otimes_I X_i \longrightarrow X_j$,

such that

- $\rho_k \iota_j = 1_{X_j}$ if $j = k$ and $0_{X_j X_k}$ otherwise.
- $\sum_{i \in I} \iota_i \rho_i = 1_{\otimes_I X_i}$.

A Proposition

Proposition (Matricial Representation)

For $f : \otimes_J X_j \longrightarrow \otimes_I Y_i$, there exists a unique family $\{f_{ij}\}_{i\in I, j\in J} : X_j \longrightarrow Y_i$ with $f = \sum_{i\in I, j\in J} \iota_i f_{ij} \rho_j$, namely, $f_{ij} = \rho_i f \iota_j$.

In particular, for $|I| = m, |J| = n$

$$f = \begin{bmatrix} f_{11} & \dots & f_{1n} \\ \vdots & \vdots & \vdots \\ f_{m1} & \dots & f_{mn} \end{bmatrix}$$

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  $\rho_j(x, i)$ is undefined for $i \neq j$ and $\rho_j(x, j) = x$,
- $\iota_j : X_j \longrightarrow \otimes_{i \in I} X_i$ by $\iota_j(x) = (x, j)$.

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- $\iota_j : X_j \longrightarrow \otimes_{i \in I} X_i$,
  $\iota_j = \{(x, (x, j)) \mid x \in X_j\} = \rho_j^{op}$.

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- $\ell_2(X)$: the set of all complex valued functions a on X for which the (unordered) sum $\sum_{x \in X} |a(x)|^2$ is finite.
- $\ell_2(X)$ is a Hilbert space
- $||a|| = (\sum_{x \in X} |a(x)|^2)^{1/2}$
- $< a, b >= \sum_{x \in X} a(x)\overline{b(x)}$ for $a, b \in \ell_2(X)$

- Barr's ℓ_2 functor: contravariant faithful functor

$$\ell_2 : PInj^{op} \longrightarrow Hilb$$

where Hilb is the category of Hilbert spaces and linear contractions (norm ≤ 1).

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2. Given $f : X \longrightarrow Y$ in PInj, $\ell_2(f) : \ell_2(Y) \longrightarrow \ell_2(X)$ is defined by

$$\ell_2(f)(b)(x) = \begin{cases} b(f(x)) & \text{if } x \in Dom(f), \\ 0 & \text{otherwise.} \end{cases}$$

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- $\ell_2(X \times Y) \cong \ell_2(X) \otimes \ell_2(Y)$
- $\ell_2(X \uplus Y) \cong \ell_2(X) \oplus \ell_2(Y)$

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- For $\ell_2(X)$ and $\ell_2(Y)$ in Hilb_2, the Hilbert space tensor product $\ell_2(X) \otimes \ell_2(Y)$ yields a tensor product in Hilb_2.

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- For $\ell_2(X)$ and $\ell_2(Y)$ in Hilb_2, the Hilbert space tensor product $\ell_2(X) \otimes \ell_2(Y)$ yields a tensor product in Hilb_2.
- Similarly for $\ell_2(X)$ and $\ell_2(Y)$ in Hilb_2, the direct sum $\ell_2(X) \oplus \ell_2(Y)$ yields a tensor product (*not* a coproduct) in Hilb_2.

The structure on PInj makes Hilb_2 into a UDC.

- $\{\ell_2(f_i)\}_I \in \mathrm{Hilb}_2(\ell_2(X), \ell_2(Y))$, $\{f_i\} \in \mathrm{PInj}(Y, X)$, $\{\ell_2(f_i)\}$ is summable if $\{f_i\}$ is summable in PInj
- $\sum_i \ell_2(f_i) \stackrel{def}{=} \ell_2(\sum_i f_i)$.

Categorical trace (JSV 96)

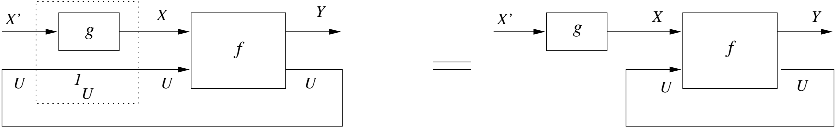
Definition

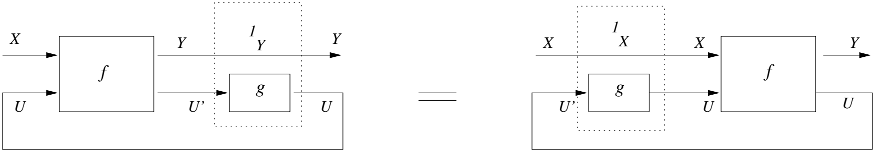
A *traced symmetric monoidal category* is a symmetric monoidal category $(\mathbb{C}, \otimes, I, s)$ with a family of functions $Tr^U_{X,Y} : \mathbb{C}(X \otimes U, Y \otimes U) \longrightarrow \mathbb{C}(X, Y)$ called a *trace*, subject to the following axioms:

- **Natural** in X, $Tr^U_{X,Y}(f)g = Tr^U_{X',Y}(f(g \otimes 1_U))$ where $f : X \otimes U \longrightarrow Y \otimes U$, $g : X' \longrightarrow X$,
- **Natural** in Y, $gTr^U_{X,Y}(f) = Tr^U_{X,Y'}((g \otimes 1_U)f)$ where $f : X \otimes U \longrightarrow Y \otimes U$, $g : Y \longrightarrow Y'$,

- **Dinatural** in U, $Tr^{U}_{X,Y}((1_Y \otimes g)f) = Tr^{U'}_{X,Y}(f(1_X \otimes g))$ where $f : X \otimes U \longrightarrow Y \otimes U'$, $g : U' \longrightarrow U$,
- **Vanishing (I,II)**, $Tr^{I}_{X,Y}(f) = f$ and $Tr^{U \otimes V}_{X,Y}(g) = Tr^{U}_{X,Y}(Tr^{V}_{X \otimes U, Y \otimes U}(g))$ for $f : X \otimes I \longrightarrow Y \otimes I$ and $g : X \otimes U \otimes V \longrightarrow Y \otimes U \otimes V$,
- **Superposing**, $Tr^{U}_{X,Y}(f) \otimes g = Tr^{U}_{X \otimes W, Y \otimes Z}((1_Y \otimes s_{U,Z})(f \otimes g)(1_X \otimes s_{W,U}))$ for $f : X \otimes U \longrightarrow Y \otimes U$ and $g : W \longrightarrow Z$,
- **Yanking**, $Tr^{U}_{U,U}(s_{U,U}) = 1_U$.

Graphical Representation

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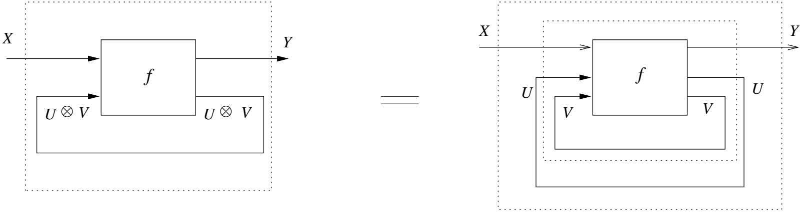
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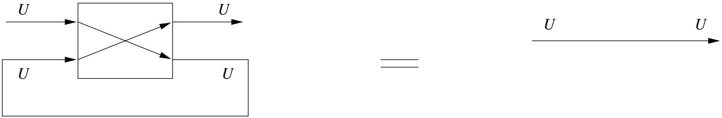
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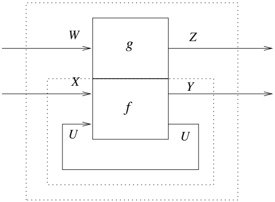
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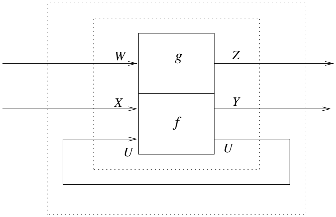
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- This is just summing $dim(U)$ many diagonal blocks, each of size $dim(W) \times dim(V)$
- See what happens when $dim(V) = dim(W) = 1$, that is when $V \cong W \cong k$

Examples, cont'd

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- This is not a product, nor a coproduct.
- Given $R : X \otimes U \longrightarrow Y \otimes U$,
  $Tr^U_{X,Y}(R) : X \longrightarrow Y$ is defined by

$$(x, y) \in Tr(R) \text{ iff } \exists u.(x, u, y, u) \in R.$$

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- Fixed Point and Iteration theory: Hasegawa, Haghverdi
- Cyclic Lambda Calculus: Hasegawa

On Ubiquity of Trace

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- Knot Theory: Jones, Joyal, Street, Freyd, Yetter
- Dimension theory of C^*-categories: Longo, Roberts
- Action Calculi: Milner and Mifsud
- Fixed Point and Iteration theory: Hasegawa, Haghverdi
- Cyclic Lambda Calculus: Hasegawa
- Asynchrony, Data flow networks: Selinger, Panangaden

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- Geometry of Interaction: Abramsky, Haghverdi
- Models of MLL: Haghverdi

Traced UDCs

Proposition (Standard Trace Formula)

Let $\mathbb{C}$ be a unique decomposition category such that for every X, Y, U and $f : X \otimes U \longrightarrow Y \otimes U$, the sum $f_{11} + \sum_{n=0}^{\infty} f_{12} f_{22}^{n} f_{21}$ exists, where f_{ij} are the components of f. Then, $\mathbb{C}$ is traced and

$$Tr_{X,Y}^{U}(f) = f_{11} + \sum_{n=0}^{\infty} f_{12} f_{22}^{n} f_{21}.$$

- Note that a UDC can be traced with a trace different from the standard one.

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- Note that a UDC can be traced with a trace different from the standard one.
- In all my work, all traced UDCs are the ones with the standard trace.

Examples: calculating traces

Let $\mathbb{C}$ be a traced UDC. Then given any $f : X \otimes U \longrightarrow Y \otimes U$, $Tr^U_{X,Y}(f)$ exists.

- Let $f : X \otimes U \longrightarrow Y \otimes U$ be given by $\begin{bmatrix} g & 0 \\ h & 0 \end{bmatrix}$. Then $Tr^U_{X,Y}(f) = Tr^U_{X,Y}\left(\begin{bmatrix} g & 0 \\ h & 0 \end{bmatrix}\right) = g + \sum_n 00^n h = g + 0h = g + 0 = g$.
- Let $f : X \otimes U \longrightarrow Y \otimes U$ be given by $\begin{bmatrix} g & 0 \\ 0 & h \end{bmatrix}$. Then

$$Tr^U_{X,Y}(f) = Tr^U_{X,Y}\left(\begin{bmatrix} g & 0 \\ 0 & h \end{bmatrix}\right) = g + \sum_n 0h^n 0 = g + 0 = g.$$

GoI Situation

Definition

A *GoI Situation* is a triple $(\mathbb{C}, T, U)$ where:

- $\mathbb{C}$ is a TSMC, Not necessarily a traced UDC!

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- $T : \mathbb{C} \longrightarrow \mathbb{C}$ is a traced symmetric monoidal functor with the following retractions:
  1. $TT \lhd T \ (e, e')$ (Comultiplication)
  2. $Id \lhd T \ (d, d')$ (Dereliction)
  3. $T \otimes T \lhd T \ (c, c')$ (Contraction)
  4. $\mathcal{K}_I \lhd T \ (w, w')$ (Weakening).

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  4. $\mathcal{K}_I \lhd T \ (w, w')$ (Weakening).
- U a reflexive object of $\mathbb{C}$:
  1. $U \otimes U \lhd U \ (j, k)$
  2. $I \lhd U$
  3. $TU \lhd U \ (u, v)$

Example: PInj

- In PInj we let $\otimes = \uplus$,
- The tensor unit is the empty set $\emptyset$.
- $T = \mathbb{N} \times -$, with $T = (T, \psi, \psi_I)$:
  $\psi_{X,Y} : \mathbb{N} \times X \uplus \mathbb{N} \times Y \longrightarrow \mathbb{N} \times (X \uplus Y)$ given by
  $(1, (n, x)) \mapsto (n, (1, x))$ and $(2, (n, y)) \mapsto (n, (2, y))$.
  ψ has an inverse defined by: $(n, (1, x)) \mapsto (1, (n, x))$ and
  $(n, (2, y)) \mapsto (2, (n, y))$.
  $\psi_I : \emptyset \longrightarrow \mathbb{N} \times \emptyset$ given by $1_\emptyset$.

- T is additive, and thus it is also traced:
  Given $f : X \uplus U \longrightarrow Y \uplus U$:
  $1_{\mathbb{N}} \times Tr^{U}_{X,Y}(f) = Tr^{\mathbb{N}\times U}_{\mathbb{N}\times X,\mathbb{N}\times Y}(\psi^{-1}(1_{\mathbb{N}} \times f)\psi)$.
- $\mathbb{N}$ is a reflexive object.
  1. $\mathbb{N} \uplus \mathbb{N} \lhd \mathbb{N}(j, k)$ is given as follows:
     $j : \mathbb{N} \uplus \mathbb{N} \longrightarrow \mathbb{N}, j(1, n) = 2n, j(2, n) = 2n + 1$ and
     $k : \mathbb{N} \longrightarrow \mathbb{N} \uplus \mathbb{N}, k(n) = (1, n/2)$ for n even, and $(2, (n - 1)/2)$ for n odd.

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  2. $\emptyset \lhd \mathbb{N}$ using the empty partial function as the retract morphisms.
  3. $\mathbb{N} \times \mathbb{N} \lhd \mathbb{N}(u, v)$ is defined as:
     $u(m, n) = <m, n> = \frac{(m+n+1)(m+n)}{2} + n$ (Cantor surjective pairing) and v as its inverse, $v(n) = (n_1, n_2)$ with $<n_1, n_2> = n$.

PInj cont'd

We next define the necessary monoidal natural transformations.

- $\mathbb{N} \times (\mathbb{N} \times X) \xrightarrow{e_X} \mathbb{N} \times X$ and $\mathbb{N} \times X \xrightarrow{e'_X} \mathbb{N} \times (\mathbb{N} \times X)$

PInj cont'd

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- $\mathbb{N} \times (\mathbb{N} \times X) \xrightarrow{e_X} \mathbb{N} \times X$ is defined by,
  $e_X(n_1, (n_2, x)) = (<n_1, n_2>, x)$.

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- $X \xrightarrow{d_X} \mathbb{N} \times X$ and $\mathbb{N} \times X \xrightarrow{d'_X} X$ $d_X(x) = (n_0, x)$ for a fixed $n_0 \in \mathbb{N}$.

PInj cont'd

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- $\mathbb{N} \times (\mathbb{N} \times X) \xrightarrow{e_X} \mathbb{N} \times X$ and $\mathbb{N} \times X \xrightarrow{e'_X} \mathbb{N} \times (\mathbb{N} \times X)$
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- $X \xrightarrow{d_X} \mathbb{N} \times X$ and $\mathbb{N} \times X \xrightarrow{d'_X} X$
  $d_X(x) = (n_0, x)$ for a fixed $n_0 \in \mathbb{N}$.
-
$$d'_X(n, x) = \begin{cases} x, & \text{if } n = n_0; \\ \text{undefined}, & \text{else.} \end{cases}$$

- $(\mathbb{N} \times X) \uplus (\mathbb{N} \times X) \xrightarrow{c_X} \mathbb{N} \times X$ and $\mathbb{N} \times X \xrightarrow{c'_X} (\mathbb{N} \times X) \uplus (\mathbb{N} \times X)$.

$$c_X = \begin{cases} (1,(n,x)) \mapsto (2n,x) \\ (2,(n,x)) \mapsto (2n+1,x) \end{cases}$$

$$c'_X(n,x) = \begin{cases} (1,(n/2,x)), & \text{if } n \text{ is even;} \\ (2,((n-1)/2,x)), & \text{if } n \text{ is odd.} \end{cases}$$

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- $\emptyset \xrightarrow{w_X} \mathbb{N} \times X$ and $\mathbb{N} \times X \xrightarrow{w'_X} \emptyset$.

Example: Traced UDC based

- $(\mathit{PInj}, \mathbb{N} \times -, \mathbb{N})$
- $(\mathit{Hilb}_2, \ell^2 \otimes -, \ell^2)$
- $(\mathit{Rel}_{\oplus}, \mathbb{N} \times -, \mathbb{N})$
- $(\mathit{Pfn}, \mathbb{N} \times -, \mathbb{N})$

GoI Interpretation

Recall that in categorical denotational semantics:

- We are given a logical system $\mathcal{L}$ to model, e.g. IL
- We are given a model category $\mathbb{C}$ with enough structure, e.g. a CCC,
- Formulas are interpreted as objects
- Proofs are intepreted as morphisms, indeed morphisms are equivalence classes of proofs
- Cut-elimination (proof transformation) is interpreted by provable equality.
- One proves a soundness theorem:

Theorem

Given a sequent $\Gamma \vdash A$ and proofs Π and Π' such that $\Pi \succ \Pi'$, then $[\![\Pi]\!] = [\![\Pi']\!] : [\![\Gamma]\!] \longrightarrow [\![A]\!]$.

GoI interpretation

In GoI interpretation:

- We are given a logical system $\mathcal{L}$ to model, e.g. MLL,
- We are given a GoI Situation $(\mathbb{C}, T, U)$, e.g. $(PInj, \mathbb{N} \times -, \mathbb{N})$,
- Formulas are interpreted as *types* (see below),
- Proofs are interpreted as morphisms in $\mathbb{C}(U, U)$,
- Cut-elimination (proof transformation) is interpreted by *the execution formula*

- One proves a finiteness theorem

Theorem

Given a sequent $\Gamma \vdash A$ *with a proof* Π *and cut formulas represented by* σ*, then* $EX(\theta(\Pi), \sigma)$ *exists.*

- One proves a finiteness theorem

Theorem

Given a sequent $\Gamma \vdash A$ with a proof Π and cut formulas represented by σ, then $EX(\theta(\Pi), \sigma)$ exists.

- And a soundness theorem

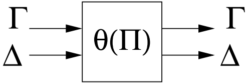
Theorem

Given a sequent $\Gamma \vdash A$ and proofs Π and Π' such that $\Pi \succ \Pi'$, then $EX(\theta(\Pi), \sigma) = EX(\theta(\Pi'), \tau)$ where σ and τ represent the cut formulas in Π and Π' respectively.

GoI Interpretation: proofs

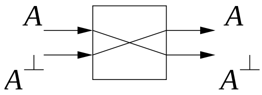
Hereafter we shall be working with traced UDCs.

- Π a proof of $\vdash [\Delta], \Gamma$, $|\Delta| = 2m$ and $|\Gamma| = n$.
- Δ keeps track of the cut formulas, e.g., $\Delta = A, A^{\perp}, B, B^{\perp}$,
- $$\theta(\Pi) : U^{n+2m} \longrightarrow U^{n+2m}$$
- $$\sigma : U^{2m} \longrightarrow U^{2m} = s_{U,U}^{\otimes m}$$



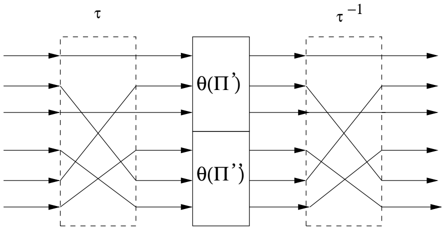
GoI Int, cont'd

axiom: $\vdash A, A^{\perp}, \quad m = 0, n = 2.$
$\theta(\Pi) = s_{U,U}.$



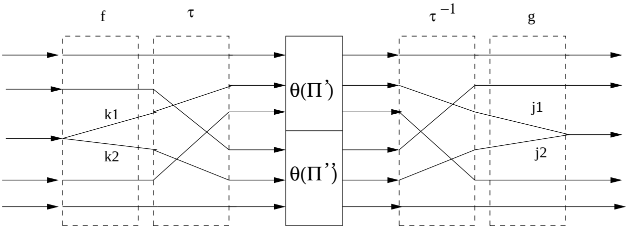
cut:

$$\dfrac{\begin{array}{c}\Pi' \\ \vdots \\ \vdash [\Delta'], \Gamma', A\end{array} \qquad \begin{array}{c}\Pi'' \\ \vdots \\ \vdash [\Delta''], A^{\perp}, \Gamma''\end{array}}{\vdash [\Delta', \Delta'', A, A^{\perp}], \Gamma', \Gamma''} \; (cut)$$



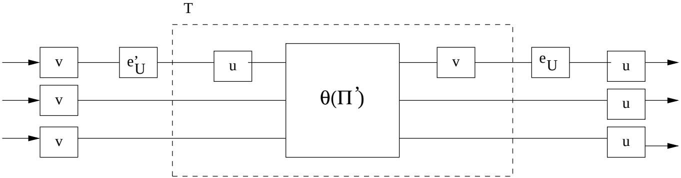
times: Recall $U \otimes U \lhd U\,(j,k)$

$$\dfrac{\begin{array}{c}\Pi' \\ \vdots \\ \vdash [\Delta'], \Gamma', A\end{array} \qquad \begin{array}{c}\Pi'' \\ \vdots \\ \vdash [\Delta''], \Gamma'', B\end{array}}{\vdash [\Delta', \Delta''], \Gamma', \Gamma'', A \otimes B}\ (\mathit{times})$$



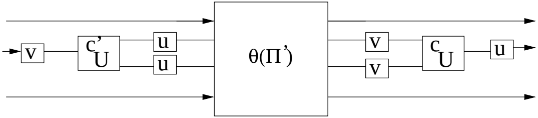
of course: Recall $TU \lhd U\,(u,v)$ and $TT \lhd T\,(e,e')$

$$\dfrac{\begin{array}{c}\Pi' \\ \vdots \\ \vdash [\Delta], ?\Gamma', A\end{array}}{\vdash [\Delta], ?\Gamma', !A}\ (\mathit{ofcourse})$$



contraction: Recall $TU \lhd U\,(u,v)$ and $T \otimes T \lhd T\,(c,c')$.

$$\dfrac{\begin{array}{c}\Pi' \\ \vdots \\ \vdash [\Delta], \Gamma', ?A, ?A\end{array}}{\vdash [\Delta], \Gamma', ?A}\ (\mathit{contraction})$$



Examples

Let Π be the following proof:

$$\frac{\vdash A, A^{\perp} \quad \vdash A, A^{\perp}}{\vdash [A^{\perp}, A], A, A^{\perp}}\ (cut)$$

Then the GoI semantics of this proof is given by

$$\theta(\Pi) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Now consider the following proof

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdash B, B^{\perp} \quad \vdash C, C^{\perp}}{\vdash B, C, B^{\perp} \otimes C^{\perp}}}{\vdash B, B^{\perp} \otimes C^{\perp}, C}}{\vdash B^{\perp} \otimes C^{\perp}, B, C}}{\vdash B^{\perp} \otimes C^{\perp}, B \parr C} .$$

Its denotation is given by

$$\begin{bmatrix} 0 & j_1 k_1 + j_2 k_2 \\ j_1 k_1 + j_2 k_2 & 0 \end{bmatrix} .$$

Orthogonality & Types

- $f, g \in \mathbb{C}(U, U)$

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- $0 \perp f$ for all $f \in \mathbb{C}(U, U)$.

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- $X \subseteq \mathbb{C}(U, U)$,

$$X^{\perp} = \{f \in \mathbb{C}(U, U) | \forall g (g \in X \Rightarrow f \perp g)\}$$

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- Definition
  A *type*: $X \subseteq \mathbb{C}(U, U)$, $X = X^{\perp\perp}$.
  - 0_{UU} belongs to every type.

GoI Int, formulas

- GoI situation $(\mathbb{C}, T, U)$. j_1, j_2, k_1, k_2 components of $U \otimes U \lhd U(j,k)$.
- $\theta(\alpha) = X$, for α atomic,
- $\theta(\alpha^\perp) = (\theta\alpha)^\perp$, for α atomic,
- $\theta(A \otimes B) = \{j_1 a k_1 + j_2 b k_2 | a \in \theta A, b \in \theta B\}^{\perp\perp}$
- $\theta(A \parr B) = \{j_1 a k_1 + j_2 b k_2 | a \in (\theta A)^\perp, b \in (\theta B)^\perp\}^\perp$
- $\theta(!A) = \{uT(a)v | a \in \theta A\}^{\perp\perp}$
- $\theta(?A) = \{uT(a)v | a \in (\theta A)^\perp\}^\perp$

GoI Int, cut-elimination

- Π a proof of $\vdash [\Delta], \Gamma$ with cut formulas in Δ

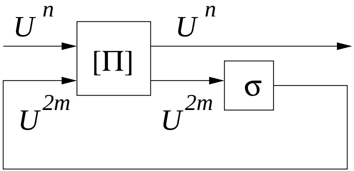
$$\begin{array}{ccc} \Pi & \rightsquigarrow & (\theta(\Pi), \sigma) \\ \\ \text{a proof of} & & \text{pair of morphisms} \\ \text{MELL} & & \text{on the object } U \end{array}$$

- execution formula $=$ standard trace formula

$\theta(\Pi) : U^{n+2m} \longrightarrow U^{n+2m}$ and $\sigma : U^{2m} \longrightarrow U^{2m}$
The dynamics is given by

$$EX(\theta(\Pi), \sigma) = Tr^{U^{2m}}_{U^n, U^n}((1_{U^n} \otimes \sigma)\theta(\Pi))$$

normalisation $\leftrightarrow$ finite sum



Which in a traced UDC is:

$$EX(\theta(\Pi), \sigma) = \pi_{11} + \sum_{n \geq 0} \pi_{12}(\sigma\pi_{22})^n(\sigma\pi_{21})$$

where $\theta(\Pi) = \begin{bmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{bmatrix}$.

Example, again!

$$\frac{\vdash A, A^{\perp} \quad \vdash A, A^{\perp}}{\vdash [A^{\perp}, A], A, A^{\perp}} \qquad \sigma = s$$

$$EX(\theta(\Pi), \sigma) = Tr\left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}\right)$$

$$= Tr\left(\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \sum_{n \geq 0} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}^n \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Associativity of cut

Lemma

Let Π be a proof of $\vdash [\Gamma, \Delta], \Lambda$ and σ and τ be the morphisms representing the cut-formulas in Γ and Δ respectively. Then

$$EX(\theta(\Pi), \sigma \otimes \tau) = EX(EX(\theta(\Pi), \tau), \sigma)$$

$$= EX(EX((1 \otimes s)\theta(\Pi)(1 \otimes s), \sigma), \tau)$$

Proof.

$EX(EX(\theta(\Pi), \tau), \sigma)$

$= Tr((1 \otimes \sigma) Tr((1 \otimes \tau)\theta(\Pi)))$

$= Tr^{U^2}(Tr^{U^2}[(1 \otimes \sigma \otimes 1)(1 \otimes \tau)\theta(\Pi)])$

$= Tr^{U^4}((1 \otimes \sigma \otimes \tau)\theta(\Pi))$

$= EX(\theta(\Pi), \sigma \otimes \tau)$

The big picture

$$\begin{array}{ccc} \text{proof} & \rightsquigarrow & \text{algorithm} \\ \text{cut-elim.} \downarrow & & \downarrow \text{computation} \\ \text{cut-free proof} & \rightsquigarrow & \text{datum} \end{array}$$

$$\begin{array}{ccc} \Pi & \rightsquigarrow & \theta(\Pi) \\ \text{cut-elim.} \downarrow & & \downarrow \text{computation} \\ \Pi' & \rightsquigarrow & \theta(\Pi') = EX(\theta(\Pi), \sigma) \end{array}$$

Towards the theorems

- $\Gamma = A_1, \cdots, A_n$.
- A *datum* of type $\theta\Gamma$:
  $M : U^n \longrightarrow U^n$, for any $\beta_1 \in \theta(A_1^\perp), \cdots, \beta_n \in \theta(A_n^\perp)$,

$$(\beta_1 \otimes \cdots \otimes \beta_n) \perp M$$

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  $$(\beta_1 \otimes \cdots \otimes \beta_n) \perp M$$
- An *algorithm* of type $\theta\Gamma$:
  $M : U^{n+2m} \longrightarrow U^{n+2m}$ for some non-negative integer m, for $\sigma : U^{2m} \longrightarrow U^{2m} = s^{\otimes m}$,
  $$EX(M, \sigma) = Tr((1 \otimes \sigma)M)$$
  is a finite sum and a datum of type $\theta\Gamma$.

A Lemma

Lemma

Let $M : U^n \longrightarrow U^n$ *and* $a : U \longrightarrow U$. *Define*
$CUT(a, M) = (a \otimes 1_{U^{n-1}})M : U^n \longrightarrow U^n$.
Then $M = [m_{ij}]$ *is a datum of type* $\theta(A, \Gamma)$ *iff*

- *for any* $a \in \theta A^\perp$, $a \perp m_{11}$, *and*
- *the morphism* $ex(CUT(a, M)) = Tr^A(s^{-1}_{\Gamma,A} CUT(a, M) s_{\Gamma,A})$ *is in* $\theta(\Gamma)$.

Theorem (Convergence or Finiteness)

Let Π *be a proof of* $\vdash [\Delta], \Gamma$. *Then* $\theta(\Pi)$ *is an algorithm of type* $\theta\Gamma$.

Proof.

A taster!

Π is an axiom, where $\Gamma = A, A^{\perp}$, then we need to prove that $EX(\theta(\Pi), 0) = \theta(\Pi)$ is a datum of type $\theta\Gamma$. That is, for all $a \in \theta A^{\perp}$ and $b \in \theta A$, $M = (a \otimes b)\theta(\Pi) = \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix}$ must be nilpotent.

Observe that $M^n = \begin{bmatrix} (ab)^{n/2} & 0 \\ 0 & (ba)^{n/2} \end{bmatrix}$ for n even and $M^n = \begin{bmatrix} 0 & (ab)^{(n-1)/2}a \\ (ba)^{(n-1)/2}b & 0 \end{bmatrix}$ for n odd. But $a \perp b$ and hence ab and ba are nilpotent. Therefore M is nilpotent. □

Invariance

Theorem (Soundness)

Let Π be a proof of a sequent $\vdash [\Delta], \Gamma$ in MELL. Then

(i) $EX(\theta(\Pi), \sigma)$ *is a finite sum.*

(ii) *If Π reduces to Π' by any sequence of cut-elimination steps and Γ does not contain any formulas of the form $?A$, then $EX(\theta(\Pi), \sigma) = EX(\theta(\Pi'), \tau)$. So $EX(\theta(\Pi), \sigma)$ is an invariant of reduction. In particular, if Π' is any cut-free proof obtained from Π by cut-elimination, then $EX(\theta(\Pi), \sigma) = \theta(\Pi')$.*

Proof.

A taster Part (i) is an easy corollary of Convergence Theorem. We proceed to the proof of part (ii).

Suppose Π' is a cut-free proof of $\vdash \Gamma, A$ and Π is obtained by applying the cut rule to Π' and the axiom $\vdash A^{\perp}, A$. Then

$$EX(\theta(\Pi), \sigma) =$$

$$Tr\left((1 \otimes \sigma)\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} \pi'_{11} & \pi'_{12} & 0 & 0 \\ \pi'_{21} & \pi'_{22} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}\right)$$

$$= Tr\left(\begin{bmatrix} \pi'_{11} & 0 & \pi'_{12} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ \pi'_{21} & 0 & \pi'_{22} & 0 \end{bmatrix}\right) = \begin{bmatrix} \pi'_{11} & \pi'_{12} \\ \pi'_{21} & \pi'_{22} \end{bmatrix} = \theta(\Pi') \qquad \square$$

Back to Girard

- $(\mathbf{PInj}, \mathbb{N} \times -, \mathbb{N})$ is a GoI situation.

- Proposition

  $(\mathbf{Hilb}_2, \ell^2 \otimes -, \ell^2)$ is a GoI Situation which agrees with Girard's C^*-algebraic model, where $\ell^2 = \ell_2(\mathbb{N})$. Its structure is induced via ℓ_2 from **PInj**.

- Proposition

  Let Π be a proof of $\vdash [\Delta], \Gamma$. Then in Girard's model $\mathbf{Hilb}_2$ *above,*

$$((1-\sigma^2)\sum_{n=0}^{\infty}\theta(\Pi)(\sigma\theta(\Pi))^n(1-\sigma^2))_{n\times n} = Tr((1\otimes\tilde{\sigma})\theta(\Pi))$$

  where $(A)_{n\times n}$ is the submatrix of A consisting of the first n rows and the first n columns. $\tilde{\sigma} = s \otimes \cdots \otimes s$ (m-times.)

The mistakes GoI makes ...

Consider the following situation:

$$\frac{\vdash !A, ?A^\perp \quad \vdash !A, ?A^\perp}{\vdash [?A^\perp, !A], !A, ?A^\perp} \qquad \succ \qquad \vdash !A, ?A^\perp$$

Note that $EX(\theta(\Pi), s) = \begin{bmatrix} 0 & ((Td')e')^2 \\ (e(Td))^2 & 0 \end{bmatrix}$

but $\theta(\Pi') = \begin{bmatrix} 0 & (Td')e' \\ e(Td) & 0 \end{bmatrix}$

Future Work

- Extension to additives
- Exploiting the GoI as a semantics: Lambda calculus, PCF etc.
- GoI 4: The Feedback Equation
- GoI 5: The Hyperfinite Factor
- Connecting to logical complexity