On Categorical Models of GoI

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January 27, 2010
In this lecture

- We shall talk about the first categorical model of GoI.
- We will consider GoI 1 (Girard 1989) for MELL.
- We emphasize the notion of categorical trace.
A critique of reductionism


- Sinn/Bedeutung
  sense/denotation
- The sense constitutes the particular way in which its denotation (reference) is given to one who grasps the thought.
- \(2 + 3 = 5\)
- sense/denotation
dynamic/static
Example

\[
\frac{A \vdash A \quad A \vdash A}{A \vdash A} \quad \triangleright \quad A \vdash A
\]

- \( id_A \circ id_A = id_A \)
- More generally, \( \Pi, \Pi' \) proofs of \( \Gamma \vdash A \), \( \Pi \triangleright \Pi' \).
- Then

\[
[\Pi] = [\Pi'] : [\Gamma] \longrightarrow [A].
\]

- A static view!
- GoI offers a dynamic semantics.
- Syntax carries irrelevant information.
Where is this dynamics to be found?
Dynamics

- Where is this dynamics to be found?
- Gentzen’s cut elimination theorem
Where is this dynamics to be found?

Gentzen’s cut elimination theorem

Theorem (Cut Elimination (Hauptsatz))

(Gentzen, 1934)

If \( \Pi \) is a proof of a sequent \( \Gamma \vdash A \), then there is a proof \( \Pi' \) of the same sequent which does not use the cut rule.

\[ \begin{align*}
\Gamma & \vdash A \\
A, \Delta & \vdash B \\
\hline
\Gamma, \Delta & \vdash B
\end{align*} \quad \text{(cut rule)} \]
Girard’s Implementation (System $\mathcal{F}$)

$\Pi \leadsto (u, \sigma)$

a proof of a pair of partial
second order LL symmetries in $\mathbb{B}(\mathcal{H})$
(no additives)

Dynamics = elimination of cuts ($\sigma$) using

$$EX(u, \sigma) = (1 - \sigma^2) \sum_{n \geq 0} u(\sigma u)^n (1 - \sigma^2)$$

Theorem (Girard, 1987)

(i) If $(u, \sigma)$ is the interpretation of a proof $\Pi$ of a sequent $\vdash [\Delta], \Gamma$ then $\sigma u$ is nilpotent.

(ii) If $\Gamma$ does not use the symbols “?” or “$\exists$”, then the interpretation is sound.

strong normalisation $\iff$ nilpotency
Back to our example

\[ \vdash A, A^\perp \vdash A, A^\perp \]
\[ \vdash [A^\perp, A], A, A^\perp \quad \succ \quad \vdash A, A^\perp \]

- proofs as matrices on \( \mathcal{M}_{2m+n}(\mathbb{B}(\ell^2)) \)
- \( u = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \)
- \( \sigma = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \)
- Dynamics: \( EX(u, \sigma) = (1 - \sigma^2)(u + u\sigma u)(1 - \sigma^2) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \)

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The big picture

proof $\leadsto$ algorithm

cut-elim. $\downarrow$ $\downarrow$ computation

cut-free proof $\leadsto$ datum

\[ \Pi \leadsto \theta(\Pi) \]

cut-elim. $\downarrow$ $\downarrow$ computation

\[ \Pi' \leadsto \theta(\Pi') = EX(\theta(\Pi), \sigma) \]
Wish list!

- Connect to other areas in mathematics to draw on their techniques:
  - Knot Theory: view proofs as knots, use knot invariants.
  - Algebraic Topology: view proofs as topological spaces, use homology.

Examples abound: Mulmuley (Alg. Geometry and Geometric Invariant Theory)
J. Friedman: Grothendieck topology
Combinatorics,...

- Study **logical complexity** using GoI tools.
  - PTIME, LOGSPACE.
  - BLL (GSS), ELL and LLL (Girard).
A Brief History, *with apologies*

- Gol 2 (1988): Deadlock-free algorithms, Recursion
- Gol 3 (1995): Additives
- Gol 4 (2003): The feedback equation
- Gol 5 (2008): The hyperfinite factor
- Danos (1990): Untyped Lambda Calculus
- Danos, Regnier, Malacaria, Mackie: Path-based Semantics
- Logical complexity related work, optimal lambda reduction, etc
History, cont’d

- Abramsky (1997): GoI Situation, Abramsky’s Program
- Haghverdi (PhD, 2000): UDC based (particle style) GoI Situation and more, including path-based semantics
- Abramsky, Haghverdi and Scott (2002): GoI Situation to CA
Definition (Kuros, Higgs, Manes, Arbib, Benson)

\((M, \Sigma)\), where \(M\) is a nonempty set and \(\Sigma\) is a partial operation on countable families in \(M\). \(\{x_i\}_{i \in I}\) is summable if \(\Sigma_{i \in I} x_i\) is defined subject to:

- **Partition-Associativity:** \(\{x_i\}_{i \in I}\) and \(\{I_j\}_{j \in J}\) a countable partition of \(I\)
  \[
  \Sigma_{i \in I} x_i = \Sigma_{j \in J} (\Sigma_{i \in I_j} x_i).
  \]

- **Unary sum:** \(\Sigma_{i \in \{j\}} x_i = x_j\).
Facts about $\Sigma$-Monoids

- $\Sigma_{i \in \emptyset} x_i$ exists and is denoted by 0. It is a countable additive identity.
- Sum is commutative and associative whenever defined.
- $\Sigma_{i \in I} x_{\varphi(i)}$ is defined for any permutation $\varphi$ of $I$, whenever $\Sigma_{i \in I} x_i$ exits.
- There are no additive inverses: $x + y = 0$ implies $x = y = 0$. 
Examples

- $M = PInj(X, Y)$, the set of partial injective functions.
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- $(\sum_i f_i)(x) = \begin{cases} f_j(x) & \text{if } x \in \text{Dom}(f_j) \text{ for some } j \in I \\ \text{undefined} & \text{otherwise.} \end{cases}$
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- \( (\sum I f_i)(x) \) as above.
More examples

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- $M$ = countably complete poset, $\Sigma = \sup$. 
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A Non-example

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- Suppose $x, y, z$ are in this family, with $x \leq z, y \leq z$ and $x, y$ incomparable, then
- $x + (y + z)$ is defined but $(x + y) + z$ is not defined.
Unique Decomposition Categories (UDCs)

Definition

A *unique decomposition category* \( \mathcal{C} \) is a symmetric monoidal category where:

- Every homset is a \( \Sigma \)-Monoid
- Composition distributes over sum (careful!)

satisfying the axiom:

(A) For all \( j \in I \)

- *quasi injection*: \( \iota_j : X_j \rightarrow \bigotimes_i X_i \),
- *quasi projection*: \( \rho_j : \bigotimes_i X_i \rightarrow X_j \),

such that

- \( \rho_k \iota_j = 1_{X_j} \) if \( j = k \) and \( 0_{X_j X_k} \) otherwise.
- \( \sum_{i \in I} \iota_i \rho_i = 1_{\bigotimes_i X_i} \).
A Proposition

Proposition (Matricial Representation)

For $f : \otimes_j X_j \to \otimes_i Y_i$, there exists a unique family
\[ \{ f_{ij} \}_{i \in I, j \in J} : X_j \to Y_i \] with
\[ f = \sum_{i \in I, j \in J} \iota_i f_{ij} \rho_j, \]
namely,
\[ f_{ij} = \iota_i f \rho_j. \]

In particular, for $|I| = m$, $|J| = n$

\[ f = \begin{bmatrix} f_{11} & \cdots & f_{1n} \\ \vdots & \ddots & \vdots \\ f_{m1} & \cdots & f_{mn} \end{bmatrix} \]
Example 1

PInj, the category of sets and partial injective functions.

- $X \otimes Y = X \uplus Y$, Not a coproduct.
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  $\rho_j(x, i)$ is undefined for $i \neq j$ and $\rho_j(x, j) = x$, 

$\iota_j : X_j \longrightarrow \bigotimes_{i \in I} X_i$ by $\iota_j(x) = (x, j)$. 

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Example 1

Plnj, the category of sets and partial injective functions.

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Example 2

Rel: The category of sets and binary relations.

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  $\rho_j = \{((x,j), x) \mid x \in X_j\}$
Rel: The category of sets and binary relations.

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- $\iota_j: X_j \longrightarrow \bigotimes_{i \in I} X_i$,
  $\iota_j = \{(x, (x, j)) \mid x \in X_j\} = \rho_j^{op}$. 
Example 3: \text{Hilb}_2

Given a set $X$, $\ell^2(X)$ is a Hilbert space with

\[ \|a\| = \left( \sum_{x \in X} |a(x)|^2 \right)^{1/2} \]

inner product $\langle a, b \rangle = \sum_{x \in X} a(x)b(x)$ for $a, b \in \ell^2(X)$. 

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On Categorical Models of GoI
Example 3: Hilb$_2$

- Given a set $X$,
- $\ell_2(X)$: the set of all complex valued functions $a$ on $X$ for which the (unordered) sum $\sum_{x \in X} |a(x)|^2$ is finite.
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- $\ell_2(X)$: the set of all complex valued functions $a$ on $X$ for which the (unordered) sum $\sum_{x \in X} |a(x)|^2$ is finite.
- $\ell_2(X)$ is a Hilbert space
- $\|a\| = (\sum_{x \in X} |a(x)|^2)^{1/2}$
- $\langle a, b \rangle = \sum_{x \in X} a(x) \overline{b(x)}$ for $a, b \in \ell_2(X)$
Barr’s $\ell_2$ functor: contravariant faithful functor

$$\ell_2 : \text{PInj}^{\text{op}} \longrightarrow \text{Hilb}$$

where Hilb is the category of Hilbert spaces and linear contractions (norm $\leq 1$).
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1. For a set $X$, $\ell_2(X)$ is defined as above
2. Given $f : X \longrightarrow Y$ in $\text{PInj}$, $\ell_2(f) : \ell_2(Y) \longrightarrow \ell_2(X)$ is defined by

$$\ell_2(f)(b)(x) = \begin{cases} b(f(x)) & \text{if } x \in \text{Dom}(f), \\ 0 & \text{otherwise.} \end{cases}$$
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$\ell_2(X \uplus Y) \cong \ell_2(X) \oplus \ell_2(Y)$
Example cont’d: Defining Hilb_2

- Objects: \( \ell_2(X) \) for a set \( X \)
Example cont’d: Defining Hilb$_2$

- **Objects:** $\ell_2(X)$ for a set $X$
- **Arrows:** $u : \ell_2(X) \to \ell_2(Y)$ is of the form $\ell_2(f)$ for some partial injective function $f : Y \to X$
Example cont’d: Defining $\text{Hilb}_2$

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- **Arrows:** $u : \ell_2(X) \rightarrow \ell_2(Y)$ is of the form $\ell_2(f)$ for some partial injective function $f : Y \rightarrow X$
- **For** $\ell_2(X)$ and $\ell_2(Y)$ in $\text{Hilb}_2$, the Hilbert space tensor product $\ell_2(X) \otimes \ell_2(Y)$ yields a tensor product in $\text{Hilb}_2$. 

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**On Categorical Models of GoI**
Example cont’d: Defining Hilb$_2$

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- **Arrows:** $u : \ell_2(X) \rightarrow \ell_2(Y)$ is of the form $\ell_2(f)$ for some partial injective function $f : Y \rightarrow X$
- For $\ell_2(X)$ and $\ell_2(Y)$ in Hilb$_2$, the Hilbert space tensor product $\ell_2(X) \otimes \ell_2(Y)$ yields a tensor product in Hilb$_2$.
- Similarly, for $\ell_2(X)$ and $\ell_2(Y)$ in Hilb$_2$, the direct sum $\ell_2(X) \oplus \ell_2(Y)$ yields a tensor product (not a coproduct) in Hilb$_2$. 
The structure on $\text{PInj}$ makes $\text{Hilb}_2$ into a UDC.

- $\{\ell_2(f_i)\}_I \in \text{Hilb}_2(\ell_2(X), \ell_2(Y))$, $\{f_i\} \in \text{PInj}(Y, X)$, $\{\ell_2(f_i)\}$ is summable if $\{f_i\}$ is summable in $\text{PInj}$

- $\sum_i \ell_2(f_i) \overset{\text{def}}{=} \ell_2(\sum_i f_i)$. 
Definition

A traced symmetric monoidal category is a symmetric monoidal category \((\mathcal{C}, \otimes, I, s)\) with a family of functions \(\text{Tr}_{X,Y}^U : \mathcal{C}(X \otimes U, Y \otimes U) \rightarrow \mathcal{C}(X, Y)\) called a trace, subject to the following axioms:

- **Natural in** \(X\), \(\text{Tr}_{X,Y}^U(f)g = \text{Tr}_{X',Y}^U(f(g \otimes 1_U))\) where \(f : X \otimes U \rightarrow Y \otimes U\), \(g : X' \rightarrow X\),

- **Natural in** \(Y\), \(g\text{Tr}_{X,Y}^U(f) = \text{Tr}_{X,Y'}^U((g \otimes 1_U)f)\) where \(f : X \otimes U \rightarrow Y \otimes U\), \(g : Y \rightarrow Y'\),

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Dinatural in $U$, $\text{Tr}^U_{X,Y}((1_Y \otimes g)f) = \text{Tr}^{U'}_{X,Y}(f(1_X \otimes g))$ where $f : X \otimes U \longrightarrow Y \otimes U'$, $g : U' \longrightarrow U$,

Vanishing (I,II), $\text{Tr}^I_{X,Y}(f) = f$ and $\text{Tr}^{U \otimes V}_{X,Y}(g) = \text{Tr}^U_{X,Y}(\text{Tr}^V_{X \otimes U,Y \otimes U}(g))$ for $f : X \otimes I \longrightarrow Y \otimes I$ and $g : X \otimes U \otimes V \longrightarrow Y \otimes U \otimes V$,

Superposing, 
$\text{Tr}^U_{X,Y}(f) \otimes g = \text{Tr}^U_{X \otimes W,Y \otimes Z}((1_Y \otimes s_{U,Z})(f \otimes g)(1_X \otimes s_{W,U}))$ for $f : X \otimes U \longrightarrow Y \otimes U$ and $g : W \longrightarrow Z$,

Yanking, $\text{Tr}^U_{U,U}(s_{U,U}) = 1_U$. 

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On Categorical Models of GoI
Graphical Representation

\[ \xrightarrow{X'} g \xrightarrow{U} f \xrightarrow{U} Y \]

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Consider the category $\text{FDVect}_k$ of finite dimensional vector spaces and linear transformations.
Examples

- Consider the category $\text{FDVect}_k$ of finite dimensional vector spaces and linear transformations.

- Given $f : V \otimes U \to W \otimes U$, \{v_i\}, \{u_j\}, \{w_k\}$ bases for $V, U, W$ respectively.

- $f(v_i \otimes u_j) = \sum_{k,m} a_{km}^{ij} w_k \otimes u_m$, $\text{Tr}_V U W(f)(v_i) = \sum_{j,k} a_{kj}^{ij} w_k$.

- This is just summing $\dim(U)$ many diagonal blocks, each of size $\dim(W) \times \dim(V)$.

- See what happens when $\dim(V) = \dim(W) = 1$, that is when $V \overset{\sim}{=} W \overset{=}{=} k$. 
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- \[ \text{Tr}_{V,W}^U(f)(v_i) = \sum_{j,k} a_{ij}^{kj} w_k \]
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Given $R : X \otimes U \to Y \otimes U$, $Tr^U_{X,Y}(R) : X \to Y$ is defined by

$$(x, y) \in Tr(R) \text{ iff } \exists u.(x, u, y, u) \in R.$$
On Ubiquity of Trace

- Functional analysis and operator theory: Kadison & Ringrose
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- Fixed Point and Iteration theory: Hasegawa, Haghverdi
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- Fixed Point and Iteration theory: Hasegawa, Haghverdi
- Cyclic Lambda Calculus: Hasegawa
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- Functional analysis and operator theory: Kadison & Ringrose
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- Dimension theory of $C^*$-categories: Longo, Roberts
- Action Calculi: Milner and Mifsud
- Fixed Point and Iteration theory: Hasegawa, Haghverdi
- Cyclic Lambda Calculus: Hasegawa
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On Ubiquity of Trace

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On Ubiquity of Trace

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- Geometry of Interaction: Abramsky, Haghverdi
- Models of MLL: Haghverdi
Proposition (Standard Trace Formula)

Let $\mathcal{C}$ be a unique decomposition category such that for every $X, Y, U$ and $f : X \otimes U \rightarrow Y \otimes U$, the sum $f_{11} + \sum_{n=0}^{\infty} f_{12} f_{22}^n f_{21}$ exists, where $f_{ij}$ are the components of $f$. Then, $\mathcal{C}$ is traced and

$$\text{Tr}_{X,Y}^U(f) = f_{11} + \sum_{n=0}^{\infty} f_{12} f_{22}^n f_{21}.$$ 

Note that a UDC can be traced with a trace different from the standard one.
Proposition (Standard Trace Formula)

Let $\mathcal{C}$ be a unique decomposition category such that for every $X, Y, U$ and $f : X \otimes U \rightarrow Y \otimes U$, the sum $f_{11} + \sum_{n=0}^{\infty} f_{12} f_{22}^n f_{21}$ exists, where $f_{ij}$ are the components of $f$. Then, $\mathcal{C}$ is traced and

$$Tr_{X,Y}^U(f) = f_{11} + \sum_{n=0}^{\infty} f_{12} f_{22}^n f_{21}.$$

- Note that a UDC can be traced with a trace different from the standard one.
- In all my work, all traced UDCs are the ones with the standard trace.
Let $\mathbb{C}$ be a traced UDC. Then given any $f : X \otimes U \to Y \otimes U$, $Tr^U_{X,Y}(f)$ exists.

Let $f : X \otimes U \to Y \otimes U$ be given by $\begin{bmatrix} g & 0 \\ h & 0 \end{bmatrix}$. Then

$$Tr^U_{X,Y}(f) = Tr^U_{X,Y} \left( \begin{bmatrix} g & 0 \\ h & 0 \end{bmatrix} \right) = g + \sum_n 00^n h = g + 0h = g + 0 = g.$$ 

Let $f : X \otimes U \to Y \otimes U$ be given by $\begin{bmatrix} g & 0 \\ 0 & h \end{bmatrix}$. Then

$$Tr^U_{X,Y}(f) = Tr^U_{X,Y} \left( \begin{bmatrix} g & 0 \\ 0 & h \end{bmatrix} \right) = g + \sum_n 0h^n 0 = g + 0 = g.$$
Definition

A *Gol Situation* is a triple \((\mathbb{C}, T, U)\) where:

- \(\mathbb{C}\) is a TSMC, **Not necessarily a traced UDC!**

\(T: \mathbb{C} \to \mathbb{C}\) is a traced symmetric monoidal functor with the following retractions:

1. \(T(\epsilon, \epsilon')\) (Comultiplication)
2. \(\text{Id}_T(\delta, \delta')\) (Dereliction)
3. \(T \otimes T(\eta, \eta')\) (Contraction)
4. \(K_I(\omega, \omega')\) (Weakening).

\(U\) is a reflexive object of \(\mathbb{C}\):

1. \(U \otimes U(\mu, \mu')\)
2. \(I_U\)
3. \(T(U(\nu, \nu'))\)
Definition
A \textit{GoI Situation} is a triple \((C, T, U)\) where:

\begin{itemize}
  \item \(C\) is a TSMC, \textbf{Not necessarily a traced UDC!}
  \item \(T : C \rightarrow C\) is a traced symmetric monoidal functor with the following retractions:
    \begin{enumerate}
      \item \( TT \triangleleft T (e, e') \) (Comultiplication)
      \item \( \text{Id} \triangleleft T (d, d') \) (Dereliction)
      \item \( T \otimes T \triangleleft T (c, c') \) (Contraction)
      \item \( K_I \triangleleft T (w, w') \) (Weakening).
    \end{enumerate}
\end{itemize}
Definition

A *GoI Situation* is a triple \((\mathcal{C}, T, U)\) where:

- \(\mathcal{C}\) is a TSMC, **Not necessarily a traced UDC!**
- \(T : \mathcal{C} \rightarrow \mathcal{C}\) is a traced symmetric monoidal functor with the following retractions:
  1. \(TT \triangleleft T (e, e')\) (Comultiplication)
  2. \(Id \triangleleft T (d, d')\) (Dereliction)
  3. \(T \otimes T \triangleleft T (c, c')\) (Contraction)
  4. \(\mathcal{K}_I \triangleleft T (w, w')\) (Weakening).

- \(U\) a reflexive object of \(\mathcal{C}\):
  1. \(U \otimes U \triangleleft U (j, k)\)
  2. \(I \triangleleft U\)
  3. \(TU \triangleleft U (u, v)\)
In Plnj we let $\otimes = \emptyset$, 

The tensor unit is the empty set $\emptyset$.

$T = \mathbb{N} \times -$, with $T = (T, \psi, \psi_I)$:

\[
\psi_{X,Y} : \mathbb{N} \times X \cup \mathbb{N} \times Y \rightarrow \mathbb{N} \times (X \cup Y)
\]
given by $(1, (n, x)) \mapsto (n, (1, x))$ and $(2, (n, y)) \mapsto (n, (2, y))$. 

$\psi$ has an inverse defined by: $(n, (1, x)) \mapsto (1, (n, x))$ and $(n, (2, y)) \mapsto (2, (n, y))$. 

$\psi_I : \emptyset \rightarrow \mathbb{N} \times \emptyset$ given by $1_{\emptyset}$. 

This example illustrates the categorical model of GoI using tensor products and tensor units.
- $T$ is additive, and thus it is also traced:
  Given $f : X \sqcup U \to Y \sqcup U$:
  
  $1_N \times Tr_{X,Y}^U(f) = Tr_{N \times X, N \times Y}^N(\psi^{-1}(1_N \times f)\psi)$.

- $N$ is a reflexive object.
  
  1. $N \sqcup N \triangleleft N(j, k)$ is given as follows:
     
     $j : N \sqcup N \to N, j(1, n) = 2n, j(2, n) = 2n + 1$ and
     
     $k : N \to N \sqcup N, k(n) = (1, n/2)$ for $n$ even, and $(2, (n - 1)/2)$
     
     for $n$ odd.
- $T$ is additive, and thus it is also traced:
  Given $f : X \sqcup U \to Y \sqcup U$:
  
  $1_\mathbb{N} \times Tr^U_{X,Y}(f) = Tr^U_{\mathbb{N} \times X, \mathbb{N} \times Y}(\psi^{-1}(1_\mathbb{N} \times f)\psi)$.

- $\mathbb{N}$ is a reflexive object.

  1. $\mathbb{N} \sqcup \mathbb{N} \vdash \mathbb{N}(j, k)$ is given as follows:
     
     $j : \mathbb{N} \sqcup \mathbb{N} \to \mathbb{N}, j(1, n) = 2n, j(2, n) = 2n + 1$ and
     
     $k : \mathbb{N} \to \mathbb{N} \sqcup \mathbb{N}, k(n) = (1, n/2)$ for $n$ even, and $(2, (n - 1)/2)$ for $n$ odd.

  2. $\emptyset \sqsubset \mathbb{N}$ using the empty partial function as the retract morphisms.
\( T \) is additive, and thus it is also traced:

Given \( f : X \cup U \longrightarrow Y \cup U \):

\[
1_N \times Tr^U_{X,Y}(f) = Tr^{N \times U}_{N \times X, N \times Y}(\psi^{-1}(1_N \times f)\psi).
\]

\( N \) is a reflexive object.

1. \( N \cup N \triangleleft N(j, k) \) is given as follows:
   \( j : N \cup N \longrightarrow N, j(1, n) = 2n, j(2, n) = 2n + 1 \) and
   \( k : N \longrightarrow N \cup N, k(n) = (1, n/2) \) for \( n \) even, and \( (2, (n - 1)/2) \)
   for \( n \) odd.

2. \( \emptyset \triangleleft N \) using the empty partial function as the retract morphisms.

3. \( N \times N \triangleleft N(u, v) \) is defined as:
   \( u(m, n) = <m, n> = \frac{(m+n+1)(m+n)}{2} + n \) (Cantor surjective pairing) and \( v \) as its inverse, \( v(n) = (n_1, n_2) \) with
   \( <n_1, n_2> = n \).
We next define the necessary monoidal natural transformations.

\[ \mathbb{N} \times (\mathbb{N} \times X) \xrightarrow{e_X} \mathbb{N} \times X \quad \text{and} \quad \mathbb{N} \times X \xrightarrow{e'_X} \mathbb{N} \times (\mathbb{N} \times X) \]
We next define the necessary monoidal natural transformations.

- \( \mathbb{N} \times (\mathbb{N} \times X) \xrightarrow{e_X} \mathbb{N} \times X \) and \( \mathbb{N} \times X \xrightarrow{e'_X} \mathbb{N} \times (\mathbb{N} \times X) \)

- \( \mathbb{N} \times (\mathbb{N} \times X) \xrightarrow{e_X} \mathbb{N} \times X \) is defined by,
  \[ e_X(n_1, (n_2, x)) = (\langle n_1, n_2 \rangle, x). \]
We next define the necessary monoidal natural transformations.

- $\mathbb{N} \times (\mathbb{N} \times X) \xrightarrow{e_X} \mathbb{N} \times X$ and $\mathbb{N} \times X \xrightarrow{e'_X} \mathbb{N} \times (\mathbb{N} \times X)$

- $\mathbb{N} \times (\mathbb{N} \times X) \xrightarrow{e_X} \mathbb{N} \times X$ is defined by,
  
  $e_X(n_1, (n_2, x)) = (\langle n_1, n_2 \rangle, x)$.

- $X \xrightarrow{d_X} \mathbb{N} \times X$ and $\mathbb{N} \times X \xrightarrow{d'_X} X$
  
  $d_X(x) = (n_0, x)$ for a fixed $n_0 \in \mathbb{N}$. 

Esfandiar Haghverdi  On Categorical Models of GoI
We next define the necessary monoidal natural transformations.

\[
\begin{align*}
\mathbb{N} \times (\mathbb{N} \times X) &\xrightarrow{e_X} \mathbb{N} \times X \quad \text{and} \quad \mathbb{N} \times X \xrightarrow{e'_X} \mathbb{N} \times (\mathbb{N} \times X) \\
\mathbb{N} \times (\mathbb{N} \times X) &\xrightarrow{e_X} \mathbb{N} \times X \quad \text{is defined by,} \\
e_X(n_1, (n_2, x)) &= (\langle n_1, n_2 \rangle, x).
\end{align*}
\]

\[
\begin{align*}
X &\xrightarrow{d_X} \mathbb{N} \times X \quad \text{and} \quad \mathbb{N} \times X \xrightarrow{d'_X} X \\
d_X(x) &= (n_0, x) \quad \text{for a fixed } n_0 \in \mathbb{N}.
\end{align*}
\]

\[
d'_X(n, x) = \begin{cases} 
  x, & \text{if } n = n_0; \\
  \text{undefined}, & \text{else.}
\end{cases}
\]
\[(\mathbb{N} \times X) \cup (\mathbb{N} \times X) \xrightarrow{c_X} \mathbb{N} \times X \quad \text{and} \quad \mathbb{N} \times X \xrightarrow{c'_X} (\mathbb{N} \times X) \cup (\mathbb{N} \times X)\].

\[c_X = \begin{cases} (1, (n, x)) &\mapsto (2n, x) \\ (2, (n, x)) &\mapsto (2n + 1, x) \end{cases}\]

\[c'_X(n, x) = \begin{cases} (1, (n/2, x)), & \text{if } n \text{ is even;} \\ (2, ((n - 1)/2, x)), & \text{if } n \text{ is odd.} \end{cases}\]
\((\mathbb{N} \times X) \uplus (\mathbb{N} \times X) \xrightarrow{c_X} \mathbb{N} \times X\) and
\(\mathbb{N} \times X \xrightarrow{c'_X} (\mathbb{N} \times X) \uplus (\mathbb{N} \times X)\).

\[c_X = \begin{cases} (1, (n, x)) &\mapsto (2n, x) \\ (2, (n, x)) &\mapsto (2n + 1, x) \end{cases}\]

\[c'_X(n, x) = \begin{cases} (1, (n/2, x)), & \text{if } n \text{ is even;} \\ (2, ((n - 1)/2, x)), & \text{if } n \text{ is odd.} \end{cases}\]

\(\emptyset \xrightarrow{w_X} \mathbb{N} \times X\) and \(\mathbb{N} \times X \xrightarrow{w'_X} \emptyset\).
Example: Traced UDC based

- $(PInj, \mathbb{N} \times -, \mathbb{N})$
- $(\text{Hilb}_2, \ell^2 \otimes -, \ell^2)$
- $(\text{Rel}_\oplus, \mathbb{N} \times -, \mathbb{N})$
- $(Pfn, \mathbb{N} \times -, \mathbb{N})$
Recall that in categorical denotational semantics:

- We are given a logical system $\mathcal{L}$ to model, e.g. IL
- We are given a model category $\mathcal{C}$ with enough structure, e.g. a CCC,
- Formulas are interpreted as objects
- Proofs are interpreted as morphisms, indeed morphisms are equivalence classes of proofs
- Cut-elimination (proof transformation) is interpreted by provable equality.

One proves a soundness theorem:

**Theorem**

*Given a sequent $\Gamma \vdash A$ and proofs $\Pi$ and $\Pi'$ such that $\Pi \succ \Pi'$, then $\llbracket \Pi \rrbracket = \llbracket \Pi' \rrbracket : \llbracket \Gamma \rrbracket \longrightarrow \llbracket A \rrbracket$.*
In GoI interpretation:

- We are given a logical system $\mathcal{L}$ to model, e.g. MLL,
- We are given a GoI Situation $(\mathbb{C}, T, U)$, e.g. $(\text{PlInj}, \mathbb{N} \times -, \mathbb{N})$,
- Formulas are interpreted as *types* (see below),
- Proofs are interpreted as morphisms in $\mathbb{C}(U, U)$,
- Cut-elimination (proof transformation) is interpreted by *the execution formula*
One proves a finiteness theorem

**Theorem**

*Given a sequent* $\Gamma \vdash A$ *with a proof* $\Pi$ *and cut formulas represented by* $\sigma$, *then* $EX(\theta(\Pi), \sigma)$ *exists.*
One proves a finiteness theorem

**Theorem**

*Given a sequent* \( \Gamma \vdash A \) *with a proof* \( \Pi \) *and cut formulas represented by* \( \sigma \), *then* \( EX(\theta(\Pi), \sigma) \) *exists.*

And a soundness theorem

**Theorem**

*Given a sequent* \( \Gamma \vdash A \) *and proofs* \( \Pi \) *and* \( \Pi' \) *such that* \( \Pi \succ \Pi' \), *then* \( EX(\theta(\Pi), \sigma) = EX(\theta(\Pi'), \tau) \) *where* \( \sigma \) *and* \( \tau \) *represent the cut formulas in* \( \Pi \) *and* \( \Pi' \) *respectively.*
Hereafter we shall be working with traced UDCs.

- $\Pi$ a proof of $\vdash [\Delta], \Gamma, |\Delta| = 2m$ and $|\Gamma| = n$.
- $\Delta$ keeps track of the cut formulas, e.g., $\Delta = A, A^\perp, B, B^\perp$.

\[
\theta(\Pi) : U^{n+2m} \longrightarrow U^{n+2m}
\]

\[
\sigma : U^{2m} \longrightarrow U^{2m} = s_{U,U}^m
\]
axiom: \( \vdash A, A^\perp, \quad m = 0, n = 2. \)
\( \theta(\Pi) = s_{U,U}. \)
\[ \text{cut:} \]

\[
\begin{align*}
\vdash [\Delta'], \Gamma', A & \quad \vdash [\Delta''], A \perp, \Gamma'' \\
\vdash [\Delta', \Delta'', A, A \perp], \Gamma', \Gamma''
\end{align*}
\] (cut)

\[ \theta(\Pi') \]
\[ \theta(\Pi'') \]
times: Recall $U \otimes U \triangleleft U (j, k)$

$$
\Pi' \quad \Pi''
\vdots \quad \vdots
\vdash [\Delta'], \Gamma', A \quad \vdash [\Delta''], \Gamma'', B
\vdash [\Delta', \Delta''], \Gamma', \Gamma'', A \otimes B
$$

(times)
of course: Recall $TU \triangleleft U(u, v)$ and $TT \triangleleft T(e, e')$

\[\Pi'\]
\[\vdash [\Delta], ?\Gamma', A\]
\[\vdash [\Delta], ?\Gamma', !A\] (ofcourse)
contraction: Recall $TU \triangleleft U(u, v)$ and $T \otimes T \triangleleft T(c, c')$.

$$\Pi'$$

$$\vdash [\Delta], \Gamma', ?A, ?A$$

$$\vdash [\Delta], \Gamma', ?A$$ (contraction)
Let $\Pi$ be the following proof:

\[
\begin{align*}
\vdash & A, A^\perp & \vdash & A, A^\perp \\
& \vdash & [A^\perp, A], A, A^\perp & 
\end{align*}
\]

(cut)

Then the GoI semantics of this proof is given by

\[
\theta(\Pi) = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0
\end{bmatrix}
\]
Now consider the following proof

\[
\begin{align*}
& \vdash B, B^\perp \quad \vdash C, C^\perp \\
& \quad \vdash B, C, B^\perp \otimes C^\perp \\
& \quad \vdash B, B^\perp \otimes C^\perp, C \\
& \quad \vdash B^\perp \otimes C^\perp, B, C \\
& \quad \vdash B^\perp \otimes C^\perp, B \not\Rightarrow C .
\end{align*}
\]

Its denotation is given by

\[
\begin{bmatrix}
0 & j_1k_1 + j_2k_2 \\
j_1k_1 + j_2k_2 & 0
\end{bmatrix}.
\]
Orthogonality & Types

\[ f, g \in \mathbb{C}(U, U) \]
- $f, g \in \mathbb{C}(U, U)$
- $f$ is nilpotent if $\exists k \geq 1. f^k = 0$. 

Definition

A type: $X \subseteq \mathbb{C}(U, U), X = X \perp \perp$. 

0, $UU$ belongs to every type.
Orthogonality & Types

- $f, g \in \mathbb{C}(U, U)$
- $f$ is nilpotent if $\exists k \geq 1. f^k = 0$.
- $f \perp g$ if $gf$ is nilpotent.
Orthogonality & Types

- $f, g \in \mathbb{C}(U, U)$
- $f$ is nilpotent if $\exists k \geq 1. f^k = 0$.
- $f \perp g$ if $gf$ is nilpotent.
- $0 \perp f$ for all $f \in \mathbb{C}(U, U)$. 
Orthogonality & Types

- $f, g \in \mathbb{C}(U, U)$
- $f$ is nilpotent if $\exists k \geq 1. f^k = 0$.
- $f \perp g$ if $gf$ is nilpotent.
- $0 \perp f$ for all $f \in \mathbb{C}(U, U)$.
- $X \subseteq \mathbb{C}(U, U)$,

$$X_{\perp} = \{ f \in \mathbb{C}(U, U) | \forall g (g \in X \Rightarrow f \perp g) \}$$
Orthogonality & Types

- \( f, g \in \mathbb{C}(U, U) \)
- \( f \) is nilpotent if \( \exists k \geq 1. \ f^k = 0 \).
- \( f \perp g \) if \( gf \) is nilpotent.
- \( 0 \perp f \) for all \( f \in \mathbb{C}(U, U) \).
- \( X \subseteq \mathbb{C}(U, U) \),

\[
X^\perp = \{ f \in \mathbb{C}(U, U) | \forall g (g \in X \Rightarrow f \perp g) \}
\]

Definition
A type: \( X \subseteq \mathbb{C}(U, U) \), \( X = X^{\perp\perp} \).

- \( 0_{UU} \) belongs to every type.
Gol situation \((C, T, U)\). \(j_1, j_2, k_1, k_2\) components of \(U \otimes U \triangleleft U(j, k)\).

\(\theta(\alpha) = X\), for \(\alpha\) atomic,

\(\theta(\alpha^\perp) = (\theta\alpha)^\perp\), for \(\alpha\) atomic,

\(\theta(A \otimes B) = \{j_1 ak_1 + j_2 bk_2 | a \in \theta A, b \in \theta B\}^{\perp\perp}\)

\(\theta(A \bowtie B) = \{j_1 ak_1 + j_2 bk_2 | a \in (\theta A)^\perp, b \in (\theta B)^\perp\}\)

\(\theta(!A) = \{uT(a) | a \in \theta A\}^{\perp\perp}\)

\(\theta(?A) = \{uT(a) | a \in (\theta A)^\perp\}\)
GoI Int, cut-elimination

▷ Π a proof of ⊢ [Δ], Γ with cut formulas in Δ

\[ \Pi \leadsto (\theta(\Pi), \sigma) \]

a proof of pair of morphisms
MELL on the object \( U \)

▷ execution formula = standard trace formula
\[ \theta(\Pi) : U^{n+2m} \rightarrow U^{n+2m} \text{ and } \sigma : U^{2m} \rightarrow U^{2m} \]

The dynamics is given by

\[ EX(\theta(\Pi), \sigma) = Tr_{U^n, U^n}((1U^n \otimes \sigma)\theta(\Pi)) \]

normalisation \(\leftrightarrow\) finite sum
Which in a traced UDC is:

\[ EX(\theta(\Pi), \sigma) = \pi_{11} + \sum_{n \geq 0} \pi_{12}(\sigma \pi_{22})^n(\sigma \pi_{21}) \]

where \( \theta(\Pi) = \begin{bmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{bmatrix} \).
Example, again!

\[ \vdash A, A^\perp \vdash A, A^\perp \]

\[ \vdash [A^\perp, A], A, A^\perp \]

\[ EX(\theta(\Pi), \sigma) = Tr \left( \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \right) \]

\[ = Tr \left( \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \right) \]

\[ = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \sum_{n \geq 0} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \end{bmatrix}^n \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \]

\[ = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \]
Lemma

Let $\Pi$ be a proof of $\vdash [\Gamma, \Delta], \Lambda$ and $\sigma$ and $\tau$ be the morphisms representing the cut-formulas in $\Gamma$ and $\Delta$ respectively. Then

$$EX(\theta(\Pi), \sigma \otimes \tau) = EX(EX(\theta(\Pi), \tau), \sigma)$$

$$= EX(EX((1 \otimes s)\theta(\Pi)(1 \otimes s), \sigma), \tau)$$

Proof.

$$EX(EX(\theta(\Pi), \tau), \sigma)$$

$$= Tr((1 \otimes \sigma) Tr((1 \otimes \tau)\theta(\Pi)))$$

$$= Tr^U((Tr^U((1 \otimes \sigma \otimes 1)(1 \otimes \tau)\theta(\Pi))))$$

$$= Tr^U((1 \otimes \sigma \otimes \tau)\theta(\Pi))$$

$$= EX(\theta(\Pi), \sigma \otimes \tau)$$
The big picture

proof $\leadsto$ algorithm

cut-elim. $\downarrow$ $\downarrow$ computation

cut-free proof $\leadsto$ datum

$$\Pi \leadsto \theta(\Pi)$$

cut-elim. $\downarrow$ $\downarrow$ computation

$$\Pi' \leadsto \theta(\Pi') = EX(\theta(\Pi), \sigma)$$
Towards the theorems

- \( \Gamma = A_1, \ldots, A_n \).
- A *datum* of type \( \theta \Gamma \):
  \( M : U^n \rightarrow U^n \), for any \( \beta_1 \in \theta(A_1^\perp), \ldots, \beta_n \in \theta(A_n^\perp) \),
  \( (\beta_1 \otimes \cdots \otimes \beta_n) \perp M \).
Towards the theorems

- $\Gamma = A_1, \ldots, A_n$.
- A *datum* of type $\theta \Gamma$:
  
  $M : U^n \to U^n$, for any $\beta_1 \in \theta(A_1^\perp), \ldots, \beta_n \in \theta(A_n^\perp)$,

  $$(\beta_1 \otimes \cdots \otimes \beta_n) \perp M$$

- An *algorithm* of type $\theta \Gamma$:
  
  $M : U^{n+2m} \to U^{n+2m}$ for some non-negative integer $m$, for 
  
  $\sigma : U^{2m} \to U^{2m} = s^\otimes m$,

  $$EX(M, \sigma) = Tr((1 \otimes \sigma)M)$$

  is a *finite sum* and a *datum* of type $\theta \Gamma$.
Lemma

Let $M : U^n \to U^n$ and $a : U \to U$. Define

$\text{CUT}(a, M) = (a \otimes 1_{U^n} - 1)M : U^n \to U^n$.

Then $M = [m_{ij}]$ is a datum of type $\theta(A, \Gamma)$ iff

- for any $a \in \theta A^\perp$, $a \perp m_{11}$, and
- the morphism $\text{ex}(\text{CUT}(a, M)) = Tr^A(s_{\Gamma, A}^{-1} \text{CUT}(a, M)s_{\Gamma, A})$ is in $\theta(\Gamma)$. 
Theorem (Convergence or Finiteness)

Let $\Pi$ be a proof of $\vdash [\Delta], \Gamma$. Then $\theta(\Pi)$ is an algorithm of type $\theta\Gamma$. 

Proof.
A taster!

Γ is an axiom, where Γ = A, A⊥, then we need to prove that

\( EX(\theta(\Pi), 0) = \theta(\Pi) \)

is a datum of type \( \theta \Gamma \). That is, for all \( a \in \theta A⊥ \)
and \( b \in \theta A \),

\[
M = (a \otimes b)\theta(\Pi) = \begin{bmatrix}
0 & a \\
\phantom{0} & b \\
\end{bmatrix}
\]

must be nilpotent.

Observe that

\[
M^n = \begin{bmatrix}
(ab)^{n/2} & 0 \\
0 & (ba)^{n/2}
\end{bmatrix}
\]

for \( n \) even and

\[
M^n = \begin{bmatrix}
0 & (ab)^{(n-1)/2}a \\
(ba)^{(n-1)/2}b & 0
\end{bmatrix}
\]

for \( n \) odd. But \( a \perp b \) and hence \( ab \) and \( ba \) are nilpotent. Therefore \( M \) is nilpotent. \( \square \)
Invariance

**Theorem (Soundness)**

Let $\Pi$ be a proof of a sequent $\vdash [\Delta], \Gamma$ in MELL. Then

(i) $EX(\theta(\Pi), \sigma)$ is a finite sum.

(ii) If $\Pi$ reduces to $\Pi'$ by any sequence of cut-elimination steps and $\Gamma$ does not contain any formulas of the form $?A$, then $EX(\theta(\Pi), \sigma) = EX(\theta(\Pi'), \tau)$. So $EX(\theta(\Pi), \sigma)$ is an invariant of reduction. In particular, if $\Pi'$ is any cut-free proof obtained from $\Pi$ by cut-elimination, then $EX(\theta(\Pi), \sigma) = \theta(\Pi')$. 
Proof. 
A taster Part (i) is an easy corollary of Convergence Theorem. We proceed to the proof of part (ii).
Suppose $\Pi'$ is a cut-free proof of $\vdash \Gamma, A$ and $\Pi$ is obtained by applying the cut rule to $\Pi'$ and the axiom $\vdash A^\bot, A$. Then $EX(\theta(\Pi), \sigma) =$

$$Tr \left( (1 \otimes \sigma) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \pi'_{11} & \pi'_{12} & 0 & 0 \\ \pi'_{21} & \pi'_{22} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \right)$$

$$= Tr \left( \begin{pmatrix} \pi'_{11} & 0 & \pi'_{12} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ \pi'_{21} & 0 & \pi'_{22} & 0 \end{pmatrix} \right) = \begin{pmatrix} \pi'_{11} & \pi'_{12} \\ \pi'_{21} & \pi'_{22} \end{pmatrix} = \theta(\Pi')$$
(\textbf{Plnj}, \mathbb{N} \times -, \mathbb{N}) \textit{ is a } GoI \textit{ situation.}

\textbf{Proposition}

(\textbf{Hilb}_2, \ell^2 \otimes -, \ell^2) \textit{ is a } GoI \textit{ Situation which agrees with Girard’s } C^\ast\textit{-algebraic model, where } \ell^2 = \ell_2(\mathbb{N}). \textit{ Its structure is induced via } \ell_2 \textit{ from } \textbf{Plnj}.

\textbf{Proposition}

Let \( \Pi \) be a proof of \( \vdash [\Delta], \Gamma \). Then in Girard’s model \( \textbf{Hilb}_2 \) above,

\[
((1 - \sigma^2) \sum_{n=0}^{\infty} \theta(\Pi)(\sigma\theta(\Pi))^n(1 - \sigma^2))_{n \times n} = \text{Tr}((1 \otimes \bar{\sigma})\theta(\Pi))
\]

where \((A)_{n \times n}\) \textit{ is the submatrix of } A \textit{ consisting of the first } n \textit{ rows and the first } n \textit{ columns. } \bar{\sigma} = s \otimes \cdots \otimes s \textit{ (m-times.)}
Consider the following situation:

\[ \vdash !A, ?A, A \vdash \vdash !A, ?A, A \]

\[ \vdash [?A, !A], !A, ?A, A \vdash \vdash !A, ?A, A \]

Note that \( EX(\theta(\Pi), s) = \begin{bmatrix} 0 & ((Td)e')^2 \\ (e(Td))^2 & 0 \end{bmatrix} \)

but \( \theta(\Pi') = \begin{bmatrix} 0 & (Td'e') \\ e(Td) & 0 \end{bmatrix} \)
Future Work

- Extension to additives
- Exploiting the GoI as a semantics: Lambda calculus, PCF etc.
- GoI 4: The Feedback Equation
- GoI 5: The Hyperfinite Factor
- Connecting to logical complexity