

Infinite Bottles of Beer

Mathematical Concepts with Epsilon Pain

Or: A cantorial approach to Cantorian arithmetic and other mathematical melodies

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Introduction

A classic summer-camp song—at least that’s the way I think of it—goes like this:

A hundred bottles of beer on the wall, a hundred bottles of beer;
If one of those bottles should happen to fall,
99 bottles of beer on the wall.

99 bottles of beer on the wall, 99 bottles of beer;
If one of those bottles should happen to fall,
98 bottles of beer on the wall.

98 bottles of beer on the wall, 98 bottles of beer;
If one of those bottles should happen to fall...

...

One bottle of beer on the wall, one bottle of beer;
If that bottle should happen to fall,
No bottles of beer on the wall.

(The End)

The idea, of course, is to entertain kids on a long trip with a long song. For a longer trip, make a longer song by starting with 200 or 300 bottles. Some years ago, the following playful variation—for *really* long trips!—occurred to me:

Infinite bottles of beer on the wall, infinite bottles of beer;
If one of those bottles should happen to fall,
infinite bottles of beer on the wall.

Infinite bottles of beer on the wall, infinite bottles of beer;
If one of those bottles should happen to fall,
infinite bottles of beer on the wall.

(etc.)

But then I thought, why stop with countable infinities? So another version occurred to me, and another of a somewhat different kind. I started writing them down, adding contributions from friends I occasionally received. Somewhere along the way, it occurred to me that these “songs” actually carry some pedagogical value, in that they make relatively painless demonstrations of mathematical concepts that people often find hard to understand. The full collection appears below; all are by me unless stated otherwise (and “DAB” = me). Note that most of them, namely those that say “(etc.)” at the bottom, are non-terminating; in one version, the very first line is non-terminating. While the vast majority are mathematically inspired, a few are based on ideas of other types. I have no doubt many more entertaining and illuminating variations are possible. Perhaps you’d like to contribute one or more?

A last (and particularly light-hearted) comment. Some might consider these “bottles of beer” songs to be unsuitable for students below drinking age. Perhaps so, but Larry Lesser (personal communication, September 2010) quotes “mathemagician” Art Benjamin as suggesting that this can be handled by putting a radical sign over the word “beer”, thus turning it into “root beer”!

Behavior: Sequences or Functions of Time

The original infinite version is an elaboration of a finite arithmetic sequence; the below variations describe sequences or, in some cases, functions of (continuous) time. As with any sequence, these can also be viewed as step functions with (perhaps infinitely many) intervals of unit length. In these terms, the “standard” version has range as well as domain in \mathbf{N} , with its value monotonically decreasing from a finite positive value to zero in a finite number of steps: 100, to be exact. Some variations are identical up to notation, and just describe their steps and/or values in unusual ways—for example, the Euler’s identity versions (numbers 10 and 11). But many behave very differently. Some start at infinity in \mathbf{N} or \mathbf{R} ; some have rational, real or complex values that approach zero only asymptotically, if at all; some go nowhere (numbers 13 and most of 16 through 20), perhaps with a bit of oscillation (16 to 18); etc. And is the differential version (12) a step function, or does it decrease continuously from 100 to zero? Differentials obviously make sense only with continuous variables, but...

Visualization and Teaching

As always, visualization can go a long way towards making difficult ideas understandable. A few graphs on the Cartesian plane appear at the end of this article; they illustrate the original version, number 11 (Euler’s identity basic version), and number 17b (oscillating converging version). In number 17b, only part of the domain is shown, since the remainder would appear to be a straight line at $y = 100$.

It’s illuminating to imagine graphing many of the “non-standard” versions on the Cartesian or the complex plane, and students might actually enjoy doing so. (Presumably those involving complex numbers would be graphed as parametric curves on the complex plane; another possibility would be to do them as 3-D graphs.) However, it’s not easy to see how to graph quite a few of the versions—for example, the continuum-hypothesis version, those involving infinite values, and those like the Fibonacci version (32) with symbolic values. Perhaps these need visualizations other than graphs as such?

For teaching, the simple infinite version above (number 2 in the list below) might be a good place to start, depending, of course, on the class. There are several amusing ways to clarify the concept of infinity. One is comparing hotels with finite rooms, all full, and infinitely many rooms, all full, when one or infinitely many new guests arrive (Hilbert’s example).

Another is to consider what would happen if you could dump infinitely many numbered ping-pong balls into a barrel, say ten at a time, but removing one at each step in different ways, either systematically or randomly. Do this for, say, an hour, taking half as long for each step as for the preceding step; then, at the end of the hour, you've dumped infinitely many balls in and removed infinitely many—and, depending on how you decide which ones to remove, the barrel is either empty or still contains infinitely many balls. Burger and Starbird's wonderful *The Heart of Mathematics* contains an entire chapter on infinity, with a nice discussion of the ping-pong ball thought experiment. Of course, instead of ping-pong balls, one could say instead that we have infinitely many numbered beer bottles; that they fall consecutively in groups of ten at a time; but that certain bottles (perhaps one in each group) for some reason don't fall. Version 3c below is an example of this. But some of the best versions of the ping-pong-ball idea make less sense with falling beer bottles, and/or are too complex to make into a simple song.

Concepts Covered

Here's an incomplete list of the specific mathematical concepts these variations cover. Note, however, that these exact *terms* may not be used.

- infinity
- countable vs. uncountable infinities
- aleph notation for infinities
- finite and infinite sequences
- geometric sequence
- Fibonacci sequence
- step function
- Euler's identity
- derivative = rate of change
- infinitesimal
- irrational number
- complex number
- asymptotic change / convergence
- identity
- algebraic notation
- summation notation
- fixed point
- n -cycle
- Cantor set
- factorial
- Klein bottle
- roundoff error
- base 2 notation
- continuum hypothesis
- null hypothesis (H_0)
- randomness, indeterminacy

Historical Note

I'm certainly not the only person to have thought of extending the song to infinity. As of 4 May 2010, Google showed a few thousand matches total for "infinite bottles of beer on the wall" and "aleph-null bottles of beer on the wall", and a few dozen for "aleph-one bottles of beer on the wall". In fact, an aleph-null version has actually been published before (Renteln and Dundes, 2005).

Another way to extend the song to infinite length is simply to not stop at zero. This has been described as a version intended to annoy listeners, with the title "Negative Bottles of Beer on the Wall" (Wikipedia, 2011).

It should also be pointed out that the conventional version of the song is very often started from 99 instead of 100.

The “Bottles of Beer” Variations Collection

NB 1: For variations involving (expression)ⁿ, I assume the first verse has $n = 1$.

NB 2: Once a variation reaches a fixed point—and many of these quickly do so—of course one could repeat the last verse indefinitely, but it could also be argued that the song should end. To draw attention to this, I’ve marked these versions “Fixed Point” instead of “etc.” Along the same lines, one variation is a 2-cycle, and I’ve marked it as such.

1. “Annoying” version (“Negative Bottles of Beer on the Wall”)

A hundred bottles of beer on the wall, a hundred bottles of beer;
If one of those bottles should happen to fall,
99 bottles of beer on the wall.

99 bottles of beer on the wall, 99 bottles of beer;
If one of those bottles should happen to fall,
98 bottles of beer on the wall.

98 bottles of beer on the wall, 98 bottles of beer;
If one of those bottles should happen to fall...

...

One bottle of beer on the wall, one bottle of beer;
If that bottle should happen to fall,
zero bottles of beer on the wall.

Zero bottles of beer on the wall, zero bottles of beer;
If one of those bottles should happen to fall,
Minus one bottles of beer on the wall.
(*etc.*)

2. Basic transfinite version 1

Infinite bottles of beer on the wall, infinite bottles of beer;
If one of those bottles should happen to fall,
infinite bottles of beer on the wall.

Infinite bottles of beer on the wall, infinite bottles of beer;
If one of those bottles should happen to fall,
infinite bottles of beer on the wall.
(Fixed Point)

3. Basic transfinite version 2 (a generalization of version 1)

Infinite bottles of beer on the wall, infinite bottles of beer;
If finite bottles should happen to fall,
infinite bottles of beer on the wall.

Infinite bottles of beer on the wall, infinite bottles of beer;
If finite bottles should happen to fall,
infinite bottles of beer on the wall.
(Fixed Point)

3b. Transfinite subset version 1

Infinite bottles of beer on the wall, infinite bottles of beer;
If the odd-ordinal bottles should happen to fall,
infinite bottles of beer on the wall.

Infinite bottles of beer on the wall, infinite bottles of beer;
If the odd-ordinal bottles should happen to fall,
infinite bottles of beer on the wall.
(Fixed Point)

3c. Transfinite subset version 2

Infinite bottles of beer on the wall, infinite bottles of beer;
If nine of the first ten should happen to fall,
infinite bottles of beer on the wall.

Infinite bottles of beer on the wall, infinite bottles of beer;
If nine of the next ten should happen to fall,
infinite bottles of beer on the wall.

Infinite bottles of beer on the wall, infinite bottles of beer;
If nine of the next ten should happen to fall,
infinite bottles of beer on the wall.
(Fixed Point)

4. Larger-infinity version

Uncountable bottles of beer on the wall, uncountable bottles of beer;
If countable bottles should happen to fall,
uncountable bottles of beer on the wall.

Uncountable bottles of beer on the wall, uncountable bottles of beer;
If countable bottles should happen to fall,
uncountable bottles of beer on the wall.
(Fixed Point)

5. General transfinite version

Aleph- n bottles of beer on the wall, aleph- n bottles of beer;
If, for $m < n$, aleph- m bottles should happen to fall,
aleph- n bottles of beer on the wall.

Aleph- n bottles of beer on the wall, aleph- n bottles of beer;
If, for
 $m < n$, aleph- m bottles should happen to fall,
aleph- n bottles of beer on the wall.
(Fixed Point)

6. Transfinite and indeterminate version (by Richard Byrd)

Infinite bottles of beer on the wall, infinite bottles of beer;
If infinite bottles should happen to fall,
indeterminate bottles of beer on the wall.
(The End, or continue into the "Pure indeterminate version")

7. Pure indeterminate version

Indeterminate bottles of beer on the wall, indeterminate bottles of beer;

If any number, regardless of how described, should happen to fall, indeterminate bottles of beer on the wall.

Indeterminate bottles of beer on the wall, indeterminate bottles of beer;
If any number, regardless of how described, should happen to fall, indeterminate bottles of beer on the wall.
(Fixed Point)

8. Geometric-sequence version (starting arbitrarily with 2^{26})

67,108,864 bottles of beer on the wall, 67,108,864 bottles of beer;
If half of those bottles should happen to fall,
33,554,432 bottles of beer on the wall.

33,554,432 bottles of beer on the wall, 33,554,432 bottles of beer;
If half of those bottles should happen to fall,
16,777,216 bottles of beer on the wall.
(etc.)

*** A version closely related to (indeed, contained in) the previous one:

One bottle of beer on the wall, one bottle of beer;
If half of a bottle should happen to fall,
Half of a bottle of beer on the wall.

Half of a bottle of beer on the wall, half of a bottle of beer;
If 1/4 of a bottle should happen to fall,
1/4 of a bottle of beer on the wall.
(etc.)

9. Generalized geometric-sequence version

m^n bottles of beer on the wall, m^n bottles of beer;
If $m^n - m^{n-1}$ bottles should happen to fall,
 m^{n-1} bottles of beer on the wall.

m^{n-1} bottles of beer on the wall, m^{n-1} bottles of beer;
If $m^{n-1} - m^{n-2}$ bottles should happen to fall,
 m^{n-2} bottles of beer on the wall.
(etc.)

10. Euler's identity extreme version

$100(e^{2i\pi})$ bottles of beer on the wall, $100(e^{2i\pi})$ bottles of beer;
If $e^{2i\pi}$ bottles should happen to fall,
 $99(e^{2i\pi})$ bottles of beer on the wall.

$99(e^{2i\pi})$ bottles of beer on the wall, $99(e^{2i\pi})$ bottles of beer;
If $e^{2i\pi}$ bottles should happen to fall,
 $98(e^{2i\pi})$ bottles of beer on the wall.
(etc.)

11. Euler's identity basic version (by Richard Byrd)

100 bottles of beer on the wall, 100 bottles of beer;
If $e^{2i\pi}$ bottles should happen to fall,
99 bottles of beer on the wall.

99 bottles of beer on the wall, 99 bottles of beer;
If $e^{2i\pi}$ bottles should happen to fall,
98 bottles of beer on the wall.
(*etc.*)

12. Differential version (by Matthew Dovey)

100 bottles of beer on the wall, 100 bottles of beer;
If $dbottles/dt = -1$,
99 bottles of beer on the wall.

99 bottles of beer on the wall, 99 bottles of beer;
If $dbottles/dt = -1$,
98 bottles of beer on the wall.
(*etc.*)

13. Infinitesimal version

100 bottles of beer on the wall, 100 bottles of beer;
If ϵ bottles should happen to fall,
100 bottles of beer on the wall.

100 bottles of beer on the wall, 100 bottles of beer;
If ϵ bottles should happen to fall,
100 bottles of beer on the wall.
(*Fixed Point*)

14. Irrational-number version

3.1415926535 8979323846 2643383279 5028841971 6939937510...
If one of those bottles should happen to fall,
2.1415926535 8979323846 2643383279 5028841971 6939937510...

2.1415926535 8979323846 2643383279 5028841971 6939937510...
If one of those bottles should happen to fall,
1.1415926535 8979323846 2643383279 5028841971 6939937510...
(*etc.*)

15a. Increasing version 1 (by Scott Kim)

100 bottles of beer on the wall, 100 bottles of beer;
If one of those bottles should happen to clone,
101 bottles of beer on the wall.

101 bottles of beer on the wall, 101 bottles of beer;
If one of those bottles should happen to clone,
102 bottles of beer on the wall.
(*etc.*)

15b. Increasing version 2 (by Scott Kim, October 1997)

100 bottles of beer on the wall, 100 bottles of beer;
If minus one of those bottles should happen to fall,
101 bottles of beer on the wall.

101 bottles of beer on the wall, 101 bottles of beer;
If minus one of those bottles should happen to fall,
102 bottles of beer on the wall.

(etc.)

16. Explicit oscillating version

100 bottles of beer on the wall, 100 bottles of beer;
If one of those bottles should happen to fall,
99 bottles of beer on the wall.

99 bottles of beer on the wall, 99 bottles of beer;
If one of those bottles should happen to clone,
100 bottles of beer on the wall.

100 bottles of beer on the wall, 100 bottles of beer;
If one of those bottles should happen to fall,
99 bottles of beer on the wall.

99 bottles of beer on the wall, 99 bottles of beer;
If one of those bottles should happen to clone,
100 bottles of beer on the wall.
(2-Cycle)

17. Implicit oscillating version

100 bottles of beer on the wall, 100 bottles of beer;
If +1 of those bottles should happen to fall,
99 bottles of beer on the wall.

99 bottles of beer on the wall, 99 bottles of beer;
If -1 of those bottles should happen to fall,
100 bottles of beer on the wall.

...

If $(-1)^{n+1}$ of those bottles should happen to fall,
 $(99+1/2) - ((-1)^{n+1}/2)$ bottles of beer on the wall.
(etc.)

17b. Oscillating converging version

100 bottles of beer on the wall, 100 bottles of beer;
If 1 of those bottles should happen to fall,
99 bottles of beer on the wall.

99 bottles of beer on the wall, 99 bottles of beer;
If $-1/2$ of a bottle should happen to fall,
 $99+1/2$ bottles of beer on the wall.

...

$99+1/2$ bottles of beer on the wall, $99+1/2$ bottles of beer;
If $1/4$ of a bottle should happen to fall,
 $99+1/4$ bottles of beer on the wall.

...

x bottles of beer on the wall, x bottles of beer;
If $(-1/2)^{n-1}$ of a bottle should happen to fall,
 $x - (-1/2)^{n-1}$ bottles of beer on the wall.
(etc.)

18. Identity version 1 (author unknown)

100 bottles of beer on the wall, 100 bottles of beer;

Take one down, put it back up,
100 bottles of beer on the wall.

100 bottles of beer on the wall, 100 bottles of beer;
Take one down, put it back up,
100 bottles of beer on the wall.

(Fixed Point)

19. Identity version 2

A hundred bottles of beer on the wall, a hundred bottles of beer;
If none of those bottles should happen to fall,
A hundred bottles of beer on the wall.

A hundred bottles of beer on the wall, a hundred bottles of beer;
If none of those bottles should happen to fall,
A hundred bottles of beer on the wall.

(Fixed Point)

20. General identity version

n bottles of beer on the wall, n bottles of beer;
If none of those bottles should happen to fall,
 n bottles of beer on the wall.

n bottles of beer on the wall, n bottles of beer;
If none of those bottles should happen to fall,
 n bottles of beer on the wall.

(Fixed Point)

21. Branching version

100 bottles of beer on the wall, 100 bottles of beer;
If one of those bottles should fall or clone,
99 or 101 bottles of beer on the wall.

99 or 101 bottles of beer on the wall, 99 or 101 bottles of beer;
If one of those bottles should fall or clone,
98, 100, or 102 bottles of beer on the wall.

98, 100, or 102 bottles of beer on the wall, 98, 100, or 102 bottles of beer;
If one of those bottles should fall or clone,
97, 99, 101, or 103 bottles of beer on the wall.

(etc.)

22. Algebraic-notation version

n bottles of beer on the wall, n bottles of beer;
If m of those bottles should happen to fall,
 $n-m$ bottles of beer on the wall.

$n-m$ bottles of beer on the wall, $n-m$ bottles of beer;
If q of those bottles should happen to fall,
 $n-m-q$ bottles of beer on the wall.

$n-m-q$ bottles of beer on the wall, $n-m-q$ bottles of beer;
If $-t$ of those bottles should happen to fall,
 $n-m-q+t$ bottles of beer on the wall.

(etc.)

23. Summation-notation version

Most of the others are special cases of this version.

n bottles of beer on the wall, n bottles of beer;
If m_1 of those bottles should happen to fall,
 $n - m_1$ bottles of beer on the wall.

$n - m_1$ bottles of beer on the wall, $n - m_1$ bottles of beer;
If m_2 of those bottles should happen to fall,

$n - \sum_{k=1}^2 m_k$ bottles of beer on the wall.

$n - m_1$ bottles of beer on the wall, $n - m_1$ bottles of beer;
If m_2 of those bottles should happen to fall,

$n - \sum_{k=1}^3 m_k$ bottles of beer on the wall.
(etc.)

24. Arbitrary-dimensionality version

10^n bottles of beer in an n -dimensional arrangement, 10^n bottles of beer;
If one of those bottles should happen to fall,
 $(10^n) - 1$ bottles of beer in an n -dimensional arrangement.

$(10^n) - 1$ bottles of beer in an n -dimensional arrangement, $(10^n) - 1$ bottles of beer;
If one of those bottles should happen to fall,
 $(10^n) - 2$ bottles of beer in an n -dimensional arrangement.
(etc.)

25. Roundoff error version (2009–10)

100.0 bottles of beer on the wall, 100.0 bottles of beer;
If .99999 of those bottles should happen to fall,
99.00001 bottles of beer on the wall.

99.00001 bottles of beer on the wall, 99.00001 bottles of beer on the wall;
If .99999 of those bottles should happen to fall,
98.00002 bottles of beer on the wall.
(etc.)

26. Base 2 version (modified by DAB from a version by Scott Kim, October 1997)

100 bottles of beer on the wall, 100 bottles of beer;
If one of those bottles should happen to fall,
11 bottles of beer on the wall.
(etc.)

27. Other mathematical variations (by Scott Kim, October 1997)

Zero bottles of beer on the wall, zero bottles of beer;
If one of those bottles should happen to fall,
underflow.
(The End)

A hundred bottles of beer in a square, a hundred bottles of beer;

If one of the dimensions should happen to fall,
Ten bottles of beer in a line.
(*etc.*)

A hundred variations of a hundred bottles of beer on the wall, a hundred variations of a
hundred bottles of beer;
If one of the variations should happen to fall,
Then this variation might cease to exist.
(*The End*)

28. Fractal dimension (Cantor set) version (by Doug McKenna, Oct. 1997, revised by DAB)

n bottles of beer on the wall, n bottles of beer;
If the middle third of those bottles should fall,
 $(1/3 + 1/3) n$ bottles of beer on the wall.

$(1/3 + 1/3) n$ bottles of beer on the wall, $(1/3 + 1/3) n$ bottles of beer;
If each middle third of those bottles should fall,
 $(1/9 + 1/9 + 1/9 + 1/9) n$ bottles of beer on the wall.
(*etc.*)

29. Continuum-hypothesis version (by Noam Elkies, Jan. 2006)

Beth-1 bottles of beer on the wall, Beth-1 bottles of beer;
Take Aleph-1 down and pass them around:
There may be no bottles of beer on the wall.
(*The End*)

30. Complex-number version 1 (2008)

100 bottles of beer on the wall, 100 bottles of beer;
If i of those bottles should happen to fall,
 $100-i$ bottles of beer on the wall.

$100-i$ bottles of beer on the wall, $100-i$ bottles of beer;
If i of those bottles should happen to fall,
 $100-2i$ bottles of beer on the wall.
(*etc.*)

31. Complex-number version 2 (2009)

$100i$ bottles of beer on the wall, $100i$ bottles of beer;
If i of those bottles should happen to square,
 $99i-1$ bottles of beer on the wall.

$99i-1$ bottles of beer on the wall, $99i-1$ bottles of beer on the wall,
If i of those bottles should happen to square,
 $98i-2$ bottles of beer on the wall.
(*etc.*)

32. Fibonacci sequence version (by Doug McKenna, May 2010)

F_n bottles of beer on the wall, F_n bottles of beer;
If F_{n-2} of those bottles should fall,
 F_{n-1} bottles of beer on the wall.

F_{n-1} bottles of beer on the wall, F_{n-1} bottles of beer;
If F_{n-3} of those bottles should fall,

F_{n-2} bottles of beer on the wall.

F_{n-2} bottles of beer on the wall, F_{n-2} bottles of beer;
If F_{n-4} of those bottles should fall,
 F_{n-3} bottles of beer on the wall.
(*etc.*)

NB: For shorter trips, swap the indices in the second and third lines of each verse :) .

33. Factorial version (July 2010)

$n!$ bottles of beer on the wall, $n!$ bottles of beer;
If $(n-1)/n$ of those bottles should happen to fall,
 $(n-1)!$ bottles of beer on the wall.

$(n-1)!$ bottles of beer on the wall, $(n-1)!$ bottles of beer;
If $(n-2)/(n-1)$ of those bottles should happen to fall,
 $(n-2)!$ bottles of beer on the wall.
(*etc.*)

34. Topological fixed-point version (August 2010)

One full Klein bottle of beer on the wall, one full Klein bottle of beer;
If that bottle should happen to spill,
Half of a bottle of beer to refill.

A half-full Klein bottle of beer on the wall, a half-full Klein bottle of beer;
If that bottle should happen to spill,
Half of a bottle of beer to refill.

A half-full Klein bottle of beer on the wall, a half-full Klein bottle of beer;
If that bottle should happen to spill,
Half of a bottle of beer to refill.
(*Fixed Point*)

35. Statistical version (by Larry Lesser & DAB, September 2010)

100 bottles of beer on the wall plus or minus one bottle;
If one of those bottles should happen to fall,
Can't reject H_0 of 100 at all.

99 bottles of beer on the wall plus or minus one bottle;
If one of those bottles should happen to fall,
Can't reject H_0 of 99 at all.
(*etc.*)

36. Probabilistic fixed-point version (by DAB & Alan Sexton, May 2013)

A hundred bottles of beer on the wall, a hundred bottles of beer;
If random bottles should happen to fall,
Random bottles of beer on the wall.

Random bottles of beer on the wall, random bottles of beer;
If random bottles should happen to fall,
Random bottles of beer on the wall.

Random bottles of beer on the wall, random bottles of beer;
If random bottles should happen to fall,

Random bottles of beer on the wall.
(Fixed Point)

37. Metamathematical version (work in progress)

A hundred bottles of beer on the wall, a hundred bottles of beer;
If mathematics's foundations should happen to fall,
Who cares what happens to beer on the wall?
Or, for some readers:
Better look for a new job.
(The End)

Non-mathematical Variations

Early non-mathematical variation (by Scott Kim?)

A hundred choruses in the song, a hundred in the song;
If one of those choruses happens to end.
(The End)

Non-mathematical variations (by Scott Kim, October 1997)

A hundred bottles of beer on the wall, a hundred bottles of beer;
If the last letter should happen to fall,
A hundred bottles of beer on the wall.
(etc.)

Four lines of lyrics on the wall, four lines of lyrics;
If one of the lines should happen to fall,
(The End)

A hundred bottles of beer on the wall, a hundred bottles of beer;
If another wall should happen to be built,
A hundred bottles of beer on two walls.
(etc.)

Instrumental version (March 2006)

[no lyrics]

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Visualizations: the original song and versions 11 and 17b

